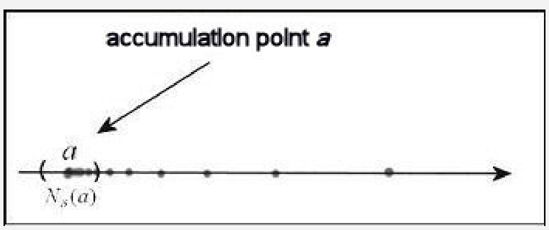
Section 5.4 Bolzano-Weierstrass and Heine-Borel Theorems

Purpose of Section: To introduce the concept of an accumulation point of a set, and state and prove two major theorems of real analysis; the Bolzano-Weierstrass Theorem and Heine-Borel Covering Theorem. Both proofs are two of the most elegant in mathematics.

Accumulation Points

Every set of real numbers has associated with it a set of accumulation limit points, a concept which allows for a precise analysis of closeness; closeness of real numbers, closeness of points in $\mathbb{R}^n$, closeness of functions, closeness of operators. The accumulation points may be a subset of a given set, part of a given set or totally disjoint of the given set. Its defining characteristic is that every accumulation point of a set is near some point of the set other than itself.

Definition: A number a is an **accumuation point** (or **limit point**) of a set A if and only if for any $\delta > 0$ the there exists the δ-neighborhood of a contains at least one point of A *distinct* from a. In other words every neighborhood of a contains points of A different from a. Keep in mind that an accumulation point of a set may or may not belong to the set.



A neighborhood of a point (any open interval containing the point) that does *not* contain the point is called a **deleted neighborhood** of the point. Thus, the set $(0,2) \sim \{1\}$ is a deleted neighborhood of 1.

Margin Note: Intuitively, an accumulation point of a set (which may or may not belong to the set) is a point where no matter how little you "wiggle" away from the point you intersect points of the set. In other words, the set likes to "snuggle up" to accumulation points.

Example 1 (Accumulation Points)

a) Every point in the closed interval $[0,1]$ is an accumulation point of the open interval $(0,1)$ since every deleted neighborhood of $a \in [0,1]$ intersects some point in $(0,1)$.

b) Finite sets have no accumulation points since around every real number (inside or outside the set) you can find a deleted neighborhood that does not contain elements of the set.

c) The set $(0,1) \cup \{2\}$ has accumulation points $[0,1]$. The number 2 is not an accumulation point of the set since there exists a deleted neighborhood around 2 that does not intersect members of the set.

d) The set $\{1/n : n = 1, 2, ...\}$ has one accumulation point at 0. Around any other point in the set you can find a deleted neighborhood that doesn't intersect the set.

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e) The integers $\mathbb{Z}$ have no accumulation point even though the set is infinite. This is easy enough to see since each integer is contained in a deleted neighborhood of radius 0.25 that does not intersect any members of the set.

Margin Note: The reader may recall limits of sequences from calculus which are examples of accumulation points of the elements in the sequence.

What do the Real Numbers *Really* Look Like?

What does the real line look like if you look at it really close up? We think of it as a continuum of points extending indefinitely in two directions, but what if you could look at it under a microscope and were able to turn up the magnification higher and higher. What would you begin to see? You might be disappointed since you will never get to a stage where you would see, "*one rational number, three irrational numbers, one rational number*, ⋯ " The real numbers are self-similar, they "look alike" no matter what the scale. So how do we "visualize" the real numbers in our minds? Well, we simply have to understand the many properties of the real numbers which can be verified mathematically, then use your imagination to visualize them it in your mind's eye.

We now come to one of the most important theorems in analysis, the Bolzano-Weierstrass theorem, but before we state and prove the theorem we must introduce ourselves to the concept of nested closed intervals.

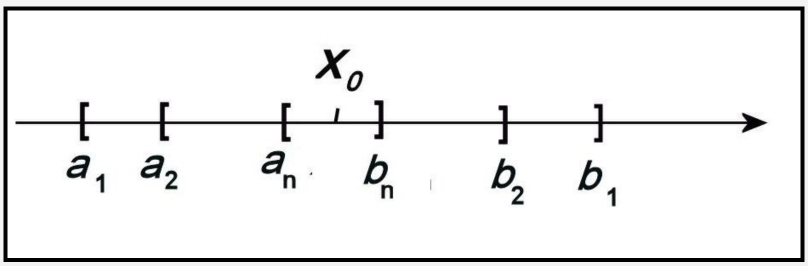
Nested Intervals

By a sequence of nested intervals $I_n = [a_n, b_n]$ we mean a sequence of closed intervals with the left endpoint a_n moving towards the right, and the right endpoint b_n moving towards the left. The question we ask is, what can be said about the intersection of all the sets; i.e. the set of all points common to every interval? The following lemma, which will be used to prove both the Bolzano-Weierstrass and Heine-Borel theorems, gives the answer.

Lemma 1 (Nested Interval Lemma)
If

$$[a_1, b_1] \supset [a_2, b_2] \supset \ldots \supset [a_n, b_n] \supset \cdots$$

is a nested sequence of closed intervals whose lengths converge to 0, i.e. $\lim_{n\to\infty}(b_n - a_n) = 0$, then their intersection consists of a single point $\bigcap_{n=1}^{\infty}[a_n, b_n] = x_0$.



Bolzano-Weierstrass Theorem

Here is an interesting question that will test your intuition about the real number system and accumulation points. Some people will answer this question in the affirmative and others in the negative, so the question is not trivial. Here is the question. Suppose you begin marking off points inside some bounded interval, open, closed, or neither, lets say $[0,1]$ for convenience, and suppose you do this indefinitely. The question is can you do it in such a way that there will *never* be an accumulation point? In other can you mark off points in such a way that they never "bunch up" anywhere? Of course it is possible to mark off points so you *do* have an accumulation point, simply pick $x_n = 1/n,\ n = 1, 2, \ldots$ which has an accumulation point at 0. In fact if you are clever, you can pick a sequence $\{x_n\}_{n=1}^{\infty}$ that has 2 accumulation points, in fact 3, 4, ⋯ , a finite number of accumulation points. It is also easy to see that if the interval is unbounded, say $[0, \infty)$ then there need not be an accumulation point, as the example $x_n = n$, $n = 1, 2, \ldots$ illustrates. So can you find a sequence $\{x_n\}_{n=1}^{\infty}$ of numbers in $[0,1]$ that does *not* have an accumulation point? Think hard.

Historical Note: Bernard Bolzano (1781-1848) was a Czech philosopher/mathematician and theologian. He was a Catholic priest who is best remembered today for his views in the methodology and mathematics and logic. Many of the ideas later developed by Cantor were understood by Bolzano. The Bolzano-Weierstrass theorem was first proven by Bolzano but unfortunately the result was lost. It was re-proven by the great German mathematician Karl Weierstrass (1815-1897). Weierstrass is often called the *father of modern analysis,* having brought mathematical rigor to the level we see today.

The answer to the question about the existence of an accumulation point of a bounded infinite set of real numbers is the statement of the Bolzano-Weierstrass theorem. The theorem is important and the proof ingenious.

Theorem 1: Bolzano-Weierstrass Theorem

Every bounded infinite set S of real numbers has an accumulation point in $\mathbb{R}$.

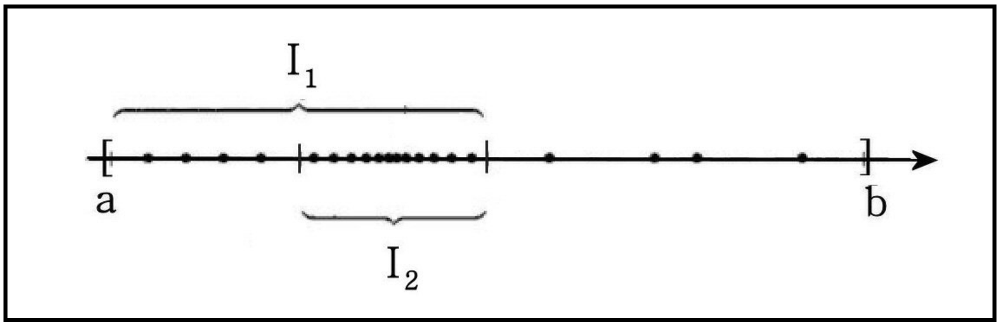
Proof:

Since S is bounded, there is a closed interval $[a,b]$ such that $S \subseteq [a,b]$, where the midpoint $x = (a+b)/2$ of $[a,b]$ divides $[a,b]$ into two closed subintervals. The subintervals overlap at the midpoint but all that matters is that their lengths are half the length of $[a,b]$. Now one of these subintervals (or possibly both) contains an infinite number of points, else the set S is the union of two finite sets, contrary to the assumption that S is infinite. Letting I_1 be the subinterval that contains an infinite number of points (if both subintervals contain an infinite number of points we pick one at random), we continue by dividing I_1 into two closed subintervals of equal length, were we call I_2 an subinterval that has contains an infinite number of points. See

Figure 1. Continuing in this manner, we arrive at a sequence of closed intervals

$$[a,b] \supset I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \cdots$$

each of whose length is half that of the previous interval. Hence by Lemma 1 the set $\bigcap_{k=1}^{\infty} I_k$ consists of a single point, say x_0.



Strategy for the Bolzano-Weierstrass Theorem; Divide and Conquer
Figure 1

We are not yet done; we must show that x_0 is an accumulation point of S. To show this let (α,β) by any neighborhood of $x_0 \in (\alpha,\beta)$, and let $k = \min(x_0-\alpha, \beta-x_0)$ and note $k>0$. Selecting an interval I_n whose length is less than k, we observe $I_n \subset (\alpha,\beta)$. But I_n contains an infinite number of points and hence so does the arbitrary neighborhood (α,β). Thus x_0 is an accumulation point of S. ∎

Margin Note: In terms of sequences, the Bolzano-Weierstrass theorem says that any bounded sequence $x_1, x_2, \ldots$ has at least one convergent subsequence.

Margin Note: Cantor claimed that the Bolzano-Weierstrass is the basis for most important results in analysis. Realize the theorem is false if one restricts oneself to infinite bounded subsets of the rational numbers. The Bolzano-Weierstrass theorem states something inherent about the real number system.

Example 2 (Accumulation Points)

a) The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ is a bounded infinite set so the Bolzano-Weierstrass theorem guarantees at least one accumulation point, which in this case there is exactly one accumulation point, namely 0. The accumulation point of this set does not belong to the set.

b) set of integers $\mathbb{Z}$ is an infinite set but is not bounded and so the conditions of the Bolzano-Weierstrass theorem are not satisfied, hence there is no guarantee of any accumulation points. In this case the set has no accumulation points.

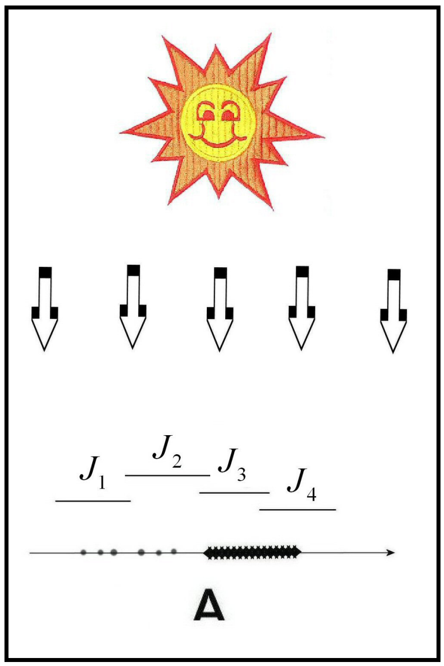
c) The set $A=(0,1)\cup\{2,3,4,...\}$ is an infinite set but not bounded so the conditions of the Bolzano-Weierstrass theorem are not satisfied; hence there is no guarantee the set has any accumulation points. However the set does have accumulation points, namely all points in the closed interval $[0,1]$.

d) The set $A=\{1,2,3,4,5\}$ is bounded but not infinite and thus the conditions of the Bolzano-Weierstrass theorem are not satisfies, hence there is no guarantee the set has any accumulation points. In this case the set does not have any accumulation points. Finite sets *never* have accumulation points.

Finite Open Covers for Sets

There are concepts in mathematics which make you ask "what does this have to do with anything," but after further study you say "wow," who ever thought of this! One such concept is the idea of coverings of sets and in particular *finite open* coverings. We will see that sets that have finite open coverings behave to a great degree like finite sets, and of course anything finite is a lot simpler than something infinite.

Definition Let A be a set of real numbers. A collection $\mathfrak{C}=\{J : J\in\mathfrak{C}\}$ of subsets of $\mathbb{R}$ is called a **cover** (or **covering**) of A if A is a subset of the union of the members of $\mathfrak{C}$, i.e. $A\subseteq\bigcup\{J : J\in\mathfrak{C}\}$. If each element $J\in\mathfrak{C}$ in the covering is an *open set* , the covering is called an **open cover**, and if $\mathfrak{C}$ contains only a *finite* number of sets, the covering $\mathfrak{C}$ is called a **finite open cover**. You might think of a covering intuitively as a collection of umbrellas providing shade from a summer sun as drawn below.



Example 3

a) The family $\mathfrak{C}=\left\{\left(0,1-\dfrac{1}{n},\right) : n\in\mathbb{N}\right\}$ is an open cover for $(0,1)$, but no finite sub-collection of these intervals will cover (0,1). Do you agree?

b) The closed bounded interval $[0,1]$ has an open cover $\mathfrak{C}=\left\{\left(-\dfrac{1}{n},1+\dfrac{1}{n},\right) : n\in\mathbb{N}\right\}$, an example being the single set $\{(-1,2)\}$ which is the member of $\mathfrak{C}$ when $n=1$.

c) The real numbers $\mathbb{R}$ is covered by the open cover $\mathfrak{C}=\{(-n,n,) : n\in\mathbb{N}\}$ but there is no finite subcover for $\mathbb{R}$.

So why is the concept of finite coverings so important in analysis? It has to do with the fact that sets which can be covered with a finite collection of open sets behave to an extent like finite sets, which of course are easier to study than infinite sets.

Compactness and the Heine-Borel Theorem

The concept of open covers of sets, in particular open covers that have finite subcovers, leads directly to one of the most important concepts in analysis, compactness.

Defintion: A set A of real numbers is called **compact** if whenever it is contained in a union of a family $\mathfrak{C}$ of open sets, it is also contained in the union of a finite number of the sets in $\mathfrak{C}$.

So now the question remains, how do we know if a set is compact; i.e. every open cover has a finite subcover? As we have seen in Example 3 some sets have this desirable property, some do not. Offhand it would seem to be a very hard property to determine. Fortunately two mathematicians, Eduard Heine and Emile Borel, found a simple characterization of these "finite-like" sets, whereby one can tell at a *glance* if they have this "finite" property. This characterization is called the Heine-Borel (Covering) Theorem.

Theorem 2 (Heine-Borel Covering Theorem)

Every closed and bounded interval $[a,b]$ is compact.

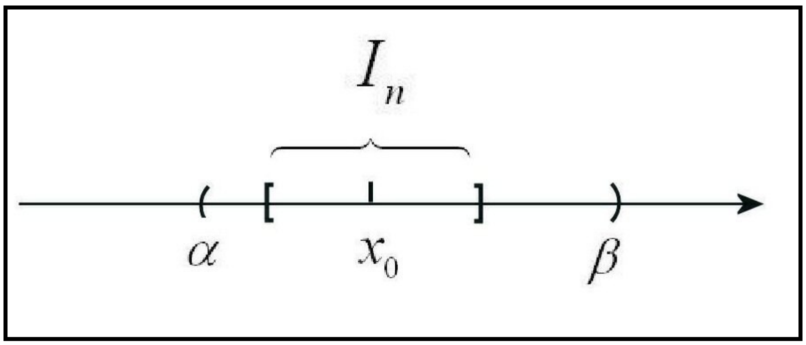
Proof:

We must prove that for any closed and bounded interval $[a,b]$ that is covered by a collection $\mathfrak{C}=\{J:J\in\mathfrak{C}\}$ of open intervals, then there is a *finite* subcollection of intervals in $\mathfrak{C}$, say $J_1,J_2,...,J_n$ which also covers $[a,b]$; that is for every $x\in[a,b]$ we have $x\in J_k$ for some $1\le k\le n$.

The proof is by contradiction. Assume the theorem false; that is, there exists an open cover of $[a,b]$ consisting of intervals $\mathfrak{C}=\{J:J\in\mathfrak{C}\}$ for which there is no finite subcover $J_1,J_2,...,J_n$. This being true, then the midpoint of $[a,b]$ divides the interval into two closed intervals, where *at least* one interval, which we call I_1, is not covered by a finite sub-collection of members of the covering $\mathfrak{C}$. We then divide I_2 in a similar manner and arrive at a new closed subinterval I_2, whose length is half that of I_2 and also is not covered by a finite number of members of the covering $\mathfrak{C}=\{J:J\in\mathfrak{C}\}$. Continuing in this manner, we arrive at a nonincreasing sequence of closed intervals

$$\left[a,b\right] \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

where each interval I_k is half as long as its predecessor and is not covered by a finite number of members of the covering $\mathfrak{C} = \{J : J \in \mathfrak{C}\}$. But from Theorem 1 we know that the intersection $\bigcap_{k=1}^{\infty} I_k$ consists of a single point, say x_0. Now since $x_0 \in [a,b]$ we know there exists (at least) one member of the family $\mathfrak{C}$ of open intervals, say (α,β) such that $x_0 \in (\alpha,\beta)$. But from the way the intervals I_k are formed there exists an interval I_n whose length is so small that $x_0 \in I_n \subseteq (\alpha,\beta) \in \mathfrak{C} = \{J : J \in \mathfrak{C}\}$.



A contradiction; we know I_n cannot be covered by a finite number of members of $\mathfrak{C}$, but it is covered by (α,β), a single element of $\mathfrak{C}$.
Figure 1

But this is a contradiction since we have said that I_n cannot be covered by a finite number of coverings of $\mathfrak{C} = \{J : J \in \mathfrak{C}\}$, but $I_n \subseteq (\alpha,\beta) \in \mathfrak{C} = \{J : J \in \mathfrak{C}\}$. Hence, we conclude every open cover of an closed and bounded interval $[a,b]$ does have a finite subcover. ∎

Margin Note: The observation that subsets of real numbers have finite covers is equivalent to being closed and bounded was first observed by German mathematician Heinrich Eduard Heine in the 1870s and later in 1894 formulated precisely by French mathematician Emile Borel.

Note on the Heine-Borel Theorem and Compact Sets

The Henie-Borel theorem as stated in Theorem 2 is a special case of a more general Heine-Borel theorem[1] which states; *A subset of* $\mathbb{R}$ *is compact iff it is closed and bounded.* We stated a specialized version of the theorem for closed intervals $[a,b]$ and we only stated the theorem one way; $[a,b] \Rightarrow \text{compactness}$ when in fact the theorem goes both ways. The Heine-Borel is important since it completely characterizes compact sets; i.e. closed and bounded sets.

Relationship Between the Bolzano-Weierstrass and Heine Borel Theorems

The Bolzano-Weierstrass and Heine-Borel Theorems are more closely related than one may think. In fact if the accumulation point(s) of the infinite set A, which is guaranteed by the Bolzao-Weierstrass theorem, belongs to A, then the set is closed and bounded, and thus by Heine-Borel have finite open covers. The following theorem, which will left unproved[2], makes this relationship precise.

Theorem 3 If A is a set of real numbers, then the following are equivalent:

i) The accumulation point(s) of A belong to A.
ii) A is closed and bounded.
iii) A is compact.

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[1] We stated that closed and bounded *intervals* have finite open subcovers whereas in fact all *closed and bounded* sets have finite subcovers. Also the converse if true; if every open cover of a set of real numbers has a finite subcover, then the set is closed and bounded.

[2] The proof of this theorem can be found in most textbooks on real analysis. A good textbook in this genera is Real Analysis by Frank Morgan, *American Mathematical Society* (2005).

Problems

1. Find the accumulation points of the following sets (if any). State whether the conditions of the Bolzano-Weierstrass theorem hold.

a) $\mathbb{N}$
b) $\mathbb{Q}$
c) $\mathbb{R}$
d) $(2,4)\cup(4,5)$
e) $\{(-1)^n : n\in\mathbb{N}\}$
f) $\varnothing$
g) $\mathbb{Q}\cap(0,1)$
h) $\left\{\frac{m}{2^n} : m,n\in\mathbb{N}\right\}$
i) $\left\{m+\frac{1}{n} : m,n\in\mathbb{N}\right\}$

2. **(Covers of Sets)** What does it mean for a family of sets not to be a cover $\mathfrak{C}$ for a set A? What does it mean for a cover $\mathfrak{C}$ of a set A not to have a finite sub-cover. Give examples of each.

3. **(Compactness)** Use the general Heine-Borel theorem, which states that a set of real numbers is compact if and only if it is closed and bounded, to determine which of the following sets are compact.

a) $\{1,2,3,4,5\}$
b) $[0,1]\cup[2,3]$
c) $\{x : x^2 = 2\}$
d) $[0,1)$
e) $[0,1]\cup\{2,3,4,5\}$

4. **(Closed Sets)** A set is closed if it contains its accumulation points. Find the accumulation points of the following sets and verify that those sets that are closed do contain their accumulation points.

a) $\mathbb{N}$
b) $\mathbb{Q}$
c) $\mathbb{R}$
d) $(2,4)\cup(4,5)$

e) $\{(-1)^n : n \in \mathbb{N}\}$
f) $\varnothing$
g) $\mathbb{Q} \cap (0,1)$
h) $\left\{\frac{m}{2^n} : m, n \in \mathbb{N}\right\}$
i) $\left\{m + \frac{1}{n} : m, n \in \mathbb{N}\right\}$

5. **(Open Subcover)** Find a finite open subcover of the set $[0,1]$ for the cover

$$\mathfrak{C} = \left\{\left(\frac{1}{j}, 1\right)_{j=1}^{\infty}\right\} \bigcup \left(-\frac{1}{10}, \frac{5}{4}\right).$$

6. **(Intersections of Closed Intervals)** The intersection of a finite number of closed intervals is one of three types of sets. What are they?

7. **(Intersections of Open Intervals)** The intersection of a finite number of open intervals is one of two types of sets. What are they?

8. **(Examples)** Give examples of the following.

a) A bounded set with no accumulation points.
b) An unbounded set with one accumulation point.
c) A set with two accumulation points.
d) An unbounded set with an infinite number of accumulation points.
e) An unbounded with one accumulation point.
f) An open set with no accumulation points.