DECOMPOSITIONS OF THE KUCHMENT-LVIN POLYNOMIALS

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THESIS ACCEPTANCE STATEMENT

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An Abstract of the Thesis Presented
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While studying the Radon transform in the context of medical tomography, Peter Kuchment and Sergey Lvin encountered an infinite family of differential polynomials $f_{n,\lambda}(u)$ where n is a positive integer, λ is a complex parameter, and u is a smooth function with derivatives $u', u'', \ldots, u^{(m)}$, etc. It was discovered that (i) if $u' = \lambda u$, then $f_{n,\lambda}(u) = 0$ for all n and that (ii) if $u'' = \lambda^2 u$, then $f_{n,\lambda}(u) = 0$ for all odd n.

This leads to a natural question (iii): if $m \geq 3$, does $u^{(m)} = \lambda^m u$ imply that $f_{n,\lambda}(u) = 0$ for a progression of n whose common difference is m?

We divide $f_{n,\lambda}(u)$ into a linear part, quadratic part, etc. according the degree of the terms. With an eye to the algebraic structure of the $f_{n,\lambda}(u)$, we then use combinatorics to prove that all parts of the polynomial vanish if $du = \lambda u$ and provide an alternate proof of identity (i). We then apply the same techniques to answer question (iii) negatively for nonzero λ and the linear part of the polynomial: If $m \geq 3$ is an integer such that $u^{(m)} = \lambda^m u$ and the linear part of $f_{n,\lambda}(u)$ vanishes for an integer $n \geq 2$, then it must be the case that $u' = \lambda u$ or $u'' = \lambda^2 u$.

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Chapter 1

Introduction

This thesis stands at the intersection of two different lines of mathematical inquiry: the study of SPECT imaging pursued by Peter Kuchment, Sergey Lvin, and others; and the push to understand differential equations by algebraic methods. One may view this thesis as a computational example in the latter context.

1.1 Identities for that came from medical imaging

The Radon transform was introduced in 1917 by Austrian mathematician Johann Radon [1]. Let $\phi(x, y)$ be a continuous, compactly supported function on $\mathbf{R}^2$ and let L denote a straight line in $\mathbf{R}^2$. The Radon transform of ϕ is the function $R\phi$ that takes L and returns the line integral of ϕ over L [2]:

$$R\phi : L \mapsto \int_L \phi \, ds.$$

The Radon transform has found applications in many diverse branches of mathematics, including partial differential equations, representation theory, and medical imaging (see [3], [4], and [5]).

In medical imaging, the practitioner seeks to recover a function $\phi(x, y)$ that describes some attribute of the body's interior (*e.g.* tissue density or distribution of medicine) [4]. The data returned by the imaging are line integrals of the desired function $\phi(x, y)$; therefore, the problem resolves to determining ϕ from its Radon transform.

In the early 1990's [5], Kuchment and Lvin studied the mathematics behind Single Photon Emission Computed Tomography (SPECT), in which the patient is given some medication weakly labeled with γ-photons [4]. Discovering the distribution of the medication resolves to studying an attenuated Radon-type transform. In checking the range conditions required to recover the original function, the authors discovered that the following identity holds for all odd positive integers n [4]:

$$\sin^n x + \sum_{k=0}^{n-1} \binom{n}{k} \left(\prod_{m=0}^{n-k-1} \left(\frac{d}{dx} - \sin x + mi \right) \right) \sin^k x = 0.$$

In 2013, the authors published "Identities for $\sin x$ that came from medical imaging" [4], which generalized this identity in the following way:

Definition 1.1.1. Let n be a positive integer, λ be a complex number, $u = u(x)$ be a smooth function, and $\partial = \frac{d}{dx}$. The n-th *Kuchment-Lvin (K-L) polynomial* parametrized by λ is defined to be

$$f_{n,\lambda}(u) = u^n + \sum_{k=0}^{n-1} \binom{n}{k} \left(\prod_{m=0}^{n-k-1} (\partial - u + m\lambda) \right) u^k.$$

Notation 1.1.2. Let α be a non-negative integer and u be a smooth function. Following notation from Singer and van der Put [6], we use ∂ for the differential operator and $u^{(\alpha)}$ for the α-th derivative of u. Specifically, $\partial^\alpha u = u^{(\alpha)}$. We will also use $u^{(0)} = u$, and $u^{(1)} = u'$, and $u^{(2)} = u''$.

In [4], the K-L polynomials were proven to vanish under the following conditions (Theorems 2 and 3 of [4] respectively):

Theorem 1.1.3 (The first Kuchment-Lvin identity)**.** *If $u = u(x)$ is a smooth function and λ is a complex number such that $u' = \lambda u$, then $f_{n,\lambda}(u) = 0$ for all $n \geq 1$.*

Theorem 1.1.4 (The second Kuchment-Lvin identity)**.** *If $u = u(x)$ is a smooth function and λ is a complex number such that $u'' = \lambda^2 u$, then $f_{n,\lambda}(u) = 0$ for all odd n.*

The results "puzzled" their discoverers [4], who had unanswered questions, particularly the following one:

Question 1.1.5. *Does some other pattern occur? Let m be an integer such that $m \geq 3$. If there exists a complex number λ such that $u^{(m)} = \lambda^m u$, then does every m-th K-L polynomial vanish?*

It is here that try to understand the K-L polynomials by another means, namely, the techniques of differential algebra.

1.2 The algebra of differential equations

In 1799, Ruffini introduced the Abel-Ruffini theorem, later proven by Abel [7], which asserted that it was not always possible to express every root of an arbitrary polynomial of degree five or greater with the basic operations of addition, multiplication, and taking roots.

Galois independently proved this result with the theory of permutations [7], leading to the development of the branch of algebra that now bears his name. His paper "Memoire sur les conditions de resolubilitie des equations par radicaux" was published posthumously by Liouville in 1846 [7].

In the early 1800's, Liouville proved [8] that second-order linear differential equations are not generally solvable by *quadrature*—in modern terms, with integrals, exponentials of integrals, or algebraic functions [9]. This was the analogue in differential equations to being unable to express the root of a polynomial by radicals.

In the early 1900's, Landau and Loewy (see [10], [11], and [12]) studied the decomposition of a differential operator, the analogue to the factorization of a polynomial. The situation is far less well-behaved because differential operators fail to commute. One must distinguish between left- and right-division, and decomposition is not necessarily unique as is the case with polynomial factorization. As an example, consider that both $\partial\partial$ and $\left(\partial + \frac{1}{x-1}\right)\left(\partial - \frac{1}{x-1}\right)$ are ways to decompose the operator ∂^2 [12]:

$$
\begin{aligned}
&\left(\partial + \frac{1}{x-1}\right)\left(\partial - \frac{1}{x-1}\right) \\
&= \partial^2 + \frac{1}{x-1}\partial - \partial\left(\frac{1}{x-1}\right) - \frac{1}{(x-1)^2} \\
&= \partial^2 + \frac{1}{x-1}\partial - \frac{1}{x-1}\partial + \frac{1}{(x-1)^2} - \frac{1}{(x-1)^2} \\
&= \partial^2.
\end{aligned}
$$

Loewy's decomposition methods can be extended to the case of partial differental equations with more sophisticated machinery [13], namely, differential algebra. The differential form of Galois theory, Picard-Vessoit theory, was introduced by Picard and Vessoit in the late 1800's and early 1900's (see [14] and [15]). The theory asserted that a linear differential equation was solvable by quadratures if and only if its differential Galois group is connected and solvable [9]. Due to a lack of understanding of Lie groups, the theory of Picard and Vessoit suffered from "a certain lack of rigor" [9] until the work of Ritt and Kolchin, the former of whom is considered the founder and "principal prophet and practioner" of the field [16].

Having only been established in the past hundred years, differential algebra is a thriving field, with applications in such diverse fields as number theory, dynamical systems, and category theory (see [17], [18], and [19] for examples).

1.3 Organization

Definition. Let λ be a complex number, n be a positive integer, u be a smooth function, and $f_{n,\lambda}(u)$ be a K-L polynomial. The *degree* of a term in $f_{n,\lambda}(u)$ is the number of derivatives of the argument being multiplied together (see Definition 2.0.6). The *linear part* $f^L_{n,\lambda}(u)$ of $f_{n,\lambda}(u)$ is the sum of all terms with degree one (see Definition 4.0.39.

This thesis will prove the following theorem in response to Question 1.0.5:

Theorem (4.0.45)**.** *Let u be a smooth function, λ be a nonzero complex number, and m be an integer such that $m \geq 3$ and $u^{(m)} = \lambda^m u$. Let n be an integer such that $n \geq 2$. If $f^L_{n,\lambda}(u) = 0$, then $u' = \lambda u$ or $u'' = \lambda^2 u$.*

In other words, for nonzero λ and the linear part of the polynomials, the answer to Question 1.0.5 is "no": all patterns that could be found are the patterns in Theorem 1.0.3 and Theorem 1.0.4.

This thesis is organized as follows:

In Chapter 2, we present definitions and lemmas that stand alone from the study of the K-L polynomials, but are necessary to the proofs given in later chapters. Particularly, we discuss coefficients that arise in differentiating a function u raised to some power (see Section 2.1), sums of products of integers (see Section 2.2), and a convolution involving a generalized binomial coefficient (see Section 2.3).

In Chapter 3, we expand the definition of the K-L polynomials in Theorem 3.0.31 and provide an alternate proof of the first Kuchment-Lvin identity by doing combinatorics on their coefficients.

Having demonstrated the utility of the combinatorial approach, in Chapter 4 we apply those techniques to prove Theorem 4.0.45 given above.

In Chapter 5, we give a more explicit proof of the case where $m = 3$ using facts about the combinatorics of the K-L polynomials and the method of integrating factors.

In Chapter 6, we describe some of the computation that was done with the Sage mathematics notebook. The emperical data provided by Sage suggested Theorems 3.0.31 and 4.0.45.

In Chapter 7, we explain the problems that arise in extending our analysis from the linear part to the whole polynomial and outline possible directions that future work on the K-L polynomials may take.

Chapter 2

Preliminaries

In Theorem 3.0.31, we express the Kuchment-Lvin polynomials (Definition 1.0.1) as a linear combination of powers of a complex number λ attached to products of derivatives of a smooth function u. Studying the K-L polynomials using combinatorics will require us to look at coefficients that arise from the product rule of differentiation and coefficients that arise from adding together certain products of integers.

2.1 Product rule coefficients

The definitions that follow depend on the choice of a smooth function u; however, we will be treating these symbols formally. For this reason we will supress u in much of the following notation. Recall from Notation 1.0.2 that if α is a non-negative integer, then $u^{(\alpha)}$ is the α-th derivative of u with $u^{(0)} = u$. In other words, $\partial^\alpha u = u^{(\alpha)}$. As usual, u' and u'' will be used in the place of $u^{(1)}$ and $u^{(2)}$ respectively. We begin by introducing some terminology to organize the terms of the K-L polynomials.

Definition 2.1.1. Let j and α be integers with $j \geq 1$ and $\alpha \geq 0$. A *differential product* π with *degree* j and *order* α is a product

$$\pi = \pi_u = \prod_{m=1}^{j} u^{(\alpha_m)}$$

such that $\alpha_1, \ldots, \alpha_j$ are non-negative integers with $\alpha_1 + \cdots + \alpha_j = \alpha$. The collection of all π with degree j and order α is denoted $\Pi_{j,\alpha} = \Pi^u_{j,\alpha}$.

We study how a differential product π can arise from differentiating powers of the function u. In particular, we are interested in coefficients of π that arise from this process.

Notation 2.1.2. Let j and α be integers with j positive. Put

$$Z_{j,\alpha} = \begin{cases} \{(\beta_1, \ldots, \beta_j) : \beta_m \geq 0 \in \mathbf{Z},\ \beta_1 + \cdots + \beta_j = \alpha\} & \alpha \geq 0 \\ \varnothing & \alpha < 0 \end{cases}$$

as the set of all tuples of j non-negative integers whose entries sum to α.

Definition 2.1.3. Let $\beta = (\beta_1, \ldots, \beta_j) \in Z_{j,\alpha}$. Let $u(x)$ be a smooth function and $\partial = \frac{d}{dx}$. The *differential word* of β with respect to u is

$$w(\beta) = w_u(\beta) = \partial^{\beta_j} u \partial^{\beta_{j-1}} u \cdots \partial^{\beta_1} u.$$

We view the above definition as function composition, read from right to left, with u playing the role of both function and operator (acting by left-multiplication). The definition for $w(\beta)$ should be read as

$$w(\beta) = \partial^{\beta_j}(u \partial^{\beta_{j-1}}(u \cdots \partial^{\beta_1} u)).$$

Example 2.1.4. Let $\beta = (0, 1, 1, 1)$. Then

$$w(\beta) = \partial u \partial u \partial u^2.$$

Evaluating this expression gives that

$$\begin{aligned}
w(\beta) =& \partial u \partial u (2uu') \\
=& \partial u \partial (2u^2 u') \\
=& \partial u (4u(u')^2 + 2u^2 u'') \\
=& \partial (4u^2 (u')^2 + 2u^3 u'') \\
=& 8u(u')^3 + 8u^2 u' u'' + 6u^2 u' u'' + 2u^3 u^{(3)} \\
=& 8u(u')^3 + 14u^2 u' u'' + 2u^3 u^{(3)}.
\end{aligned}$$

In the example above, the fact that β has a leading zero entry allows us to group the first two u's together. Let k be a positive integer. We will look at coefficients that arise while studying with k copies of u appearing before the first ∂. We introduce the following notation.

Notation 2.1.5. Let j and k be positive integers such that $k \leq j$. Let α be a non-negative integer. Denote the set of all members of $Z_{j,\alpha}$ whose first $k-1$ entries are zero by

$$Z_{j,\alpha,k} = \{\beta \in Z_{j,\alpha} : \beta_1, \ldots, \beta_{k-1} = 0\}.$$

Definition 2.1.6. Let j and k be positive integers such that $k \leq j$. Let α be a non-negative integer, let $\pi \in \Pi_{j,\alpha}$, and let $\beta \in Z_{j,\alpha,k}$. The *product rule coefficient* associated to π with respect to β is the number $P_{\beta,\pi}$ such that

$$w(\beta) = \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi} \pi.$$

Example 2.1.7. In the last example, we had $\beta = (0,1,1,1) \in Z_{4,3,2}$ and

$$w(\beta) = \partial u \partial u \partial u^2 = 8u(u')^3 + 14uu'u'' + 2u^3 u^{(3)}.$$

Above, $\pi = u(u')^3$ has the product rule coefficient 8 with respect to $\beta = (0, 1, 1, 1)$, $\pi = u^2u'u''$ has the coefficient 14, and $\pi = u^3u^{(3)}$ has the coefficient 2.

The first K-L identity (Theorem 1.0.3) supposes u satisfies the first-order linear differential equation $u' = \lambda u$ for a complex number λ. We will present definitions and lemmas that arise while studying $w(\beta)$ when $u' = \lambda u$ holds.

Lemma 2.1.8. *Let $j \geq 1$ and $\alpha \geq 0$ be integers. Suppose there exists a complex number λ such that $u' = \lambda u$. If $\pi \in \Pi_{j,\alpha}$, then $\pi = \lambda^\alpha u^j$ after repeatedly substituting $u' = \lambda u$.*

Proof. First, observe that if α_m is a positive integer and $u' = \lambda u$, then successive differentiation gives $u^{(\alpha_m)} = \lambda^{\alpha_m} u$. Next, recall π is a product of derivatives of u whose orders α_m sum to α. Then

$$\pi = \prod_{m=1}^{j} u^{(\alpha_m)} = \prod_{m=1}^{j} \lambda^{\alpha_m} u = \lambda^{\alpha_1 + \cdots + \alpha_j} \prod_{m=1}^{j} u = \lambda^\alpha u^j$$

as claimed. □

The preceding lemma allows different differential products π to be added together when $u' = \lambda u$ is assumed.

Example 2.1.9. Let $\beta = (0, 1, 1, 1)$. If λ is a complex number and $u' = \lambda u$, then

$$\begin{aligned} w(\beta) = \partial u \partial u \partial u^2 &= 8(u')^3 + 14u^2u'u'' + 2u^3u^{(3)} \\ &= 8\lambda^3u^3 + 14\lambda^3u^3 + 2\lambda^3u^3 = 24\lambda^3u^3 \end{aligned}$$

Definition 2.1.10. Let j and k be positive integers such that $k \leq j$ and let α be a non-negative integer. Let $\beta \in Z_{j,\alpha,k}$. The *density* of β is

$$|\beta| = \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi}.$$

Lemma 2.1.11. *Let j and k be positive integers such that $k \leq j$. Let α be a non-negative integer and let $\beta \in Z_{j,\alpha,k}$. If there exists a complex number λ such that $u' = \lambda u$, then*

$$w(\beta) = \left(\prod_{m=k}^{j} m^{\beta_m} \right) \lambda^{\alpha} u^j.$$

Proof. We induct on $j - k$. Let $j = k$ so that $\beta = (0, \ldots, 0, \beta_k)$ and $\beta_k = \beta_j = \alpha$. Then

$$w(\beta) = \partial^{\beta_j} u^j.$$

To show $w(\beta) = j^{\beta_j} \lambda^{\alpha} u^j$, we must perform a second induction on β_j. We establish the base case:

$$\partial^{\beta_j} = \partial u^j = j u^{j-1} u' = j \lambda u^j.$$

Suppose that $\partial^{\beta_j - 1} u^j = j^{\beta_j - 1} \lambda^{\alpha - 1} u^j$. In that case,

$$\partial^{\beta_j} u^j = \partial \partial^{\beta_j - 1} u^j = j^{\beta_j - 1} \lambda^{\alpha - 1} \partial u^j = j^{\beta_j} \lambda^{\alpha} u^j.$$

Therefore, if $\beta = (0, \ldots, 0, \beta_k)$ and $\beta_k = \beta_j = \alpha$, then

$$w(\beta) = j^{\beta_j} \lambda^{\alpha} u^j.$$

Next, suppose that the claim is true for tuples with $j - 1$ entries. Let $\beta = (0, \ldots, 0, \beta_k, \ldots, \beta_j)$ with $\beta_k + \cdots + \beta_j = \alpha$. Then

$$\begin{aligned} w(\beta) &= \partial^{\beta_j} u \partial^{\beta_{j-1}} u \cdots \partial^{\beta_k} u \\ &= \partial^{\beta_j} u \left[\left(k^{\beta_k} (k+1)^{\beta_{k+1}} \cdots (j-1)^{\beta_{j-1}} \right) \lambda^{\alpha - \beta_j} u^{j-1} \right] \\ &= \left(k^{\beta_k} (k+1)^{\beta_{k+1}} \cdots (j-1)^{\beta_{j-1}} \right) \lambda^{\alpha - \beta_j} \partial^{\beta_j} u^j. \end{aligned}$$

From the base step,

$$\partial^{\beta_j} u^j = j^{\beta_j} \lambda^{\beta_j} u^j,$$

so

$$w(\beta) = \left(\prod_{m=k}^{j} m^{\beta_m} \right) \lambda^\alpha u^j$$

as desired. □

Corollary 2.1.12. Let j and k be positive integers such that $k \leq j$. Let α be a non-negative integer and let $\beta \in Z_{j,\alpha,k}$. Then

$$|\beta| = \prod_{m=k}^{j} m^{\beta_m}.$$

Example 2.1.13. The previous example showed that if $\beta = (0, 1, 1, 1)$, then

$$|\beta| = 2^1 \cdot 3^1 \cdot 4^1 = 24.$$

Table 2.1 gives all $\beta \in Z_{4,3,2}$ and their corresponding densities. The last line of the table is the sum of the densities, which is a key component in the study of the K-L polynomials. This leads us to the following definition.

Definition 2.1.14. Let j and k be positive integers such that $k \leq j$. Let α be an integer. The *weight* of j, α, and k is

$$W(j, \alpha, k) = \begin{cases} \displaystyle\sum_{\beta \in Z^\alpha_{j,k}} |\beta| & k > 1, \alpha \geq 0 \\ W(j, \alpha, 1) & k = 0, \alpha \geq 0 \\ 0 & \alpha < 0. \end{cases}$$

$\beta \in Z_{4,3,2}$	$\sum_{\pi \in \Pi_{4,3}} P_{\beta,\pi}$
$(0,0,0,3)$	64
$(0,0,1,2)$	48
$(0,0,2,1)$	36
$(0,0,3,0)$	27
$(0,1,1,1)$	24
$(0,1,2,0)$	18
$(0,1,0,2)$	32
$(0,2,1,0)$	12
$(0,2,0,1)$	16
$(0,3,0,0)$	8
$W(4,3,2)$	285

Table 2.1: The sum of densities corresponding to $\beta \in Z_{4,3,2}$.

Remark 2.1.15. The definition of $W(j, \alpha, k)$ represents the sum of all possible product rule coefficients $P_{\beta,\pi}$ arising from

$$\partial^{\beta_j} u \cdots \partial^{\beta_k} u^k$$

for all possible choices of $\beta_k, \dots, \beta_j$. If $k > 1$ and $\alpha \geq 0$, then

$$W(j, \alpha, k) = \sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi} = \sum_{\beta \in Z_{j,\alpha,k}} |\beta|.$$

If we take $k = 0$, then the definition of $Z_{j,\alpha,k}$—recall, the set of non-negative integer j-tuples with $k-1$ leading zeroes whose entries sum to α—is nonsense. However, the definition $W(j, \alpha, 0) = W(j, \alpha, 1)$ is consistent with the above interpretation: Suppose we instead defined β to be a $(j+1)$-tuple with first entry β_0. Then we look at words of the form

$$\partial^{\beta_j} u \cdots \partial^{\beta_1} u \partial^{\beta_0} = \partial^{\beta_j} u \cdots \partial^{\beta_1} u \partial^{\beta_0} 1.$$

If $\beta_0 > 0$, then this expression is identically zero, and such a β contributes no coefficients. If $\beta_0 = 0$, we should have just considered $(\beta_1, \dots, \beta_j)$. Thus nothing is lost by restricting the discussion to j-tuples β and forcing $W(j,\alpha,0) = W(j,\alpha,1)$. Furthermore, the proof of Lemma 3.0.37 relies very critically on the fact that in Lemma 2.0.22, the index m cannot be less than one.

We have assumed that α is non-negative so far. In the proof of Lemma 3.0.37, we re-index a sum in such a way that we add terms to the where the second coordinate of $W(j,\alpha,k)$ is negative. If $\alpha < 0$, then the set $Z_{j,\alpha,k}$ is empty; there are no tuples of non-negative integers whose entries sum to α. In that case, the sum is empty and is defined to be zero.

Example 2.1.16. Table 2.1 gives that the sum of all the densities corresponding to $\beta \in Z_{4,3,2}$—*i.e.*, the weight of 4, 3, and 2—is 285. Using Corollary 2.0.17 and Definition 2.0.19, we will arrive at the same value for $W(4,3,2)$.

$$\begin{aligned} W(4,3,2) &= \sum_{\beta \in Z_{4,3,2}} 2^{\beta_2} 3^{\beta_3} 4^{\beta_4} \\ &= \sum_{\beta_2+\beta_3+\beta_4} 2^{\beta_2} 3^{\beta_3} 4^{\beta_4} \\ &= \sum_{\beta_2=0}^{3} 2^{\beta_2} \sum_{\beta_3=0}^{3-\beta_2} 3^{\beta_3} 4^{2-\beta_2-\beta_3}. \end{aligned}$$

Note that

$$\sum_{\beta_3=0}^{3-\beta_2} 3^{\beta_3} 4^{2-\beta_2-\beta_3} = 4^{3-\beta_2} \sum_{\beta_3=0}^{3-\beta_2} \left(\frac{3}{4}\right)^{\beta_3}$$

is a finite geometric series whose value is

$$\begin{aligned} 4^{3-\beta_2} \sum_{\beta_3=0}^{3-\beta_2} \left(\frac{3}{4}\right)^{\beta_3} &= 4^{3-\beta_2} \frac{1-(\frac{3}{4})^{3-\beta_2+1}}{1-\frac{3}{4}} \\ &= \frac{4^{3-\beta_2+1} - 3^{3-\beta_2+1}}{4-3}. \end{aligned}$$

Therefore,

$$\begin{aligned}W(4,3,2) = & \sum_{\beta_2=0}^{3} 2^{\beta_2}\left(4^{3-\beta_2+1} - 3^{3-\beta_2+1}\right)\\ = & \; 4\sum_{\beta_2=0}^{3} 2^{\beta_2}4^{3-\beta_2} - 3\sum_{\beta_2=0}^{3} 2^{\beta_2}3^{3-\beta_2}.\end{aligned}$$

Computing these finite geometric series in the same way, we have that

$$W(4,3,2) = \; 4\frac{4^{3+1}-2^{3+1}}{2} - 3\frac{3^{3+1}-2^{3+1}}{1} = 285.$$

The above line can be rewritten as follows:

$$W(4,3,2) = \frac{4^2}{2!}\cdot 1\cdot 4^3 - \frac{3^1}{1!}\cdot 3\cdot 3^3 + \frac{2^0}{0!}\cdot 2\cdot 2^3,$$

which takes into account the way that one would arrive at a linear combination of 4^3, 3^3, and 2^3 when computing $W(4,3,2)$. Expanding $W(j,\alpha,k)$ in this way is computationally useful in the proof of Lemma 3.0.37. Remarkably, the coefficients on 4^α, 3^α, and 2^α in these expansions do not change with α. A similar calculation shows that

$$W(4,5,2) = 6069 = \frac{4^2}{2!}\cdot 1\cdot 4^5 - \frac{3^1}{1!}\cdot 3\cdot 3^5 + \frac{2^0}{0!}\cdot 2\cdot 2^5.$$

Lemma 2.1.17. *Let j and k be positive integers such that $k \le j$. Let α be a non-negative integer. For $k \le m \le j$, there exist rational numbers $A_{m,j}$, independent of α, such that*

$$W(j,\alpha,k) = \sum_{m=k}^{j} \frac{m^{m-k}}{(m-k)!}A_{m,j}m^\alpha.$$

Proof. First, we will realize $W(j,\alpha,k)$ as a type of geometric series. Next, we will evaluate this sum to be the expression given in the lemma. Recall from Definition 2.0.19

that

$$W(j, \alpha, k) = \sum_{\beta \in Z^{\alpha}_{j,k}} |\beta|.$$

By Corollary 2.0.17,

$$|\beta| = \prod_{m=k}^{j} m^{\beta_m},$$

so

$$W(j, \alpha, k) = \sum_{\beta \in Z_{j,\alpha,k}} \prod_{m=k}^{j} m^{\beta_m}.$$

We can express this sum in the following way:

$$\begin{aligned} W(j, \alpha, k) &= \sum_{\beta \in Z_{j,\alpha,k}} \prod_{m=k}^{j} m^{\beta_m} \\ &= \sum_{\beta_k + \cdots + \beta_j = \alpha} \prod_{m=k}^{j} m^{\beta_m} \\ &= \sum_{\beta_k = 0}^{\alpha} k^{\beta_k} \sum_{\beta_{k+1} + \cdots + \beta_j = \alpha - \beta_k} \prod_{m=k+1}^{j} m^{\beta_m} \\ &= \sum_{\beta_k = 0}^{\alpha} k^{\beta_k} W(j, \alpha - \beta_k, k+1). \end{aligned}$$

This suggests induction on $j - k$ as the correct strategy. First, consider the case where $j = k$:

$$W(k, \alpha, k) = \sum_{\beta_k = \alpha} k^{\beta_k} = k^{\alpha}.$$

The base case is satisfied taking $A_{k,k} = 1$. Next, let $j > k$ and suppose the claim is true for $W(j, \alpha - \beta_k, k+1)$. Then

$$\begin{aligned} W(j, \alpha, k) &= \sum_{\beta_k=0}^{\alpha} k^{\beta_k} W(j, \alpha - \beta_k, k+1) \\ &= \sum_{\beta_k=0}^{\alpha} k^{\beta_k} \sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k-1)!} A_{m,j} m^{\alpha-\beta_k} \\ &= \sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k-1)!} A_{m,j} \sum_{\beta_k=0}^{\alpha} k^{\beta_k} m^{\alpha-\beta_k} \\ &= \sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k-1)!} A_{m,j} m^{\alpha} \sum_{\beta_k=0}^{\alpha} \left(\frac{k}{m}\right)^{\beta_k}. \end{aligned}$$

The sum

$$\sum_{\beta_k=0}^{\alpha} \left(\frac{k}{m}\right)^{\beta_k}$$

is a finite geometric series with value

$$\sum_{\beta_k=0}^{\alpha} \left(\frac{k}{m}\right)^{\beta_k} = \frac{1-(k/m)^{\alpha+1}}{1-k/m} = \frac{m - k^{\alpha+1}/m^{\alpha}}{m-k}.$$

Therefore,

$$\begin{aligned} &\sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k-1)!} A_{m,j} m^{\alpha} \frac{m - k^{\alpha+1}/m^{\alpha}}{m-k} \\ &= \sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k-1)!} A_{m,j} \frac{m^{\alpha+1} - k^{\alpha+1}}{m-k} \\ &= -\sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k)!} A_{m,j} k^{\alpha+1} + \sum_{m=k+1}^{j} \frac{m^{m-k}}{(m-k)!} A_{m,j} m^{\alpha}. \end{aligned}$$

Define

$$A_k^j = -k \sum_{m=k+1}^{j} \frac{m^{m-k-1}}{(m-k)!} A_{m,j}$$

so that

$$\begin{aligned} W(j,\alpha,k) &= A_{k,j}k^{\alpha} + \sum_{m=k+1}^{j} \frac{m^{m-k}}{(m-k)!} A_{m,j} m^{\alpha} \\ &= \frac{k^{k-k}}{(k-k)!} A_{k,j}k^{\alpha} + \sum_{m=k+1}^{j} \frac{m^{m-k}}{(m-k)!} A_{m,j} m^{\alpha} \\ &= \sum_{m=k}^{j} \frac{m^{m-k}}{m-k)!} A_{m,j} m^{\alpha} \end{aligned}$$

as needed. □

2.2 Sums of products of integers

Let n be a positive integer and α be a non-negative integer. Another key ingredient in the combinatorial proof for Kuchment and Lvin's first result is a sum of all products of α integers between 1 and n.

Definition 2.2.1. Let n and α be integers. An *α-product of n* is a product of α distinct integers between 1 and n. Let A denote a subset of $\{1,\ldots,n\}$. The sum of all such products is given by

$$S(n,\alpha) = \begin{cases} \sum_{|A|=\alpha} \prod_{a\in A} a & n \geq 1, 1 \leq \alpha \leq n \\ 1 & \alpha = 0 \\ 0 & \text{otherwise.} \end{cases}$$

These numbers can be easily understood when realized as the coefficients of the polynomial defined below.

Lemma 2.2.2. *The polynomial*

$$g_n(z) = \prod_{a=1}^{n} (1+az)$$

is a generating function for the numbers $S(n,\alpha)$. In other words,

$$g_n(z) = \sum_{\alpha=0}^{n} S(n,\alpha) z^{\alpha}.$$

Proof. Expand the product in the definition of $g_n(z)$ into a sum of powers of z. To obtain any z^{α} in the expansion of this product, we must multiply together α distinct integers a such that $1 \leq a \leq n$. The coefficient on z^{α} will be the sum of all these products, which by definition is exactly $S(n,\alpha)$. □

Remark 2.2.3. Note additionally that the only constant term is obtained by multiplying 1 together n times; $S(n,0) = 1$. This explains the definition for $S(n,\alpha)$.

Lemma 2.2.4. *Let n be a positive integer and α be a negative integer. If $1 \leq m \leq n$, then*

$$\sum_{\alpha=0}^{n} m^{n-\alpha+1} S(n,\alpha) = \frac{(n+m)!}{(m-1)!}. \tag{2.1}$$

Proof. Notice that

$$m^{n+1} g_n\left(m^{-1}\right) = m^{n+1} \sum_{\alpha=0}^{n} m^{-\alpha} S(n,\alpha) = \sum_{\alpha=0}^{n} m^{n-\alpha+1} S(n,\alpha).$$

Using the product form of $g_n(z)$ we have

$$m^{n+1} g_n\left(m^{-1}\right) = m^{n+1} \prod_{a=1}^{n} \left(1 + \frac{a}{m}\right).$$

We can multiply each factor of the product by one of the copies of m to get

$$m \prod_{a=1}^{n} (m+a)$$

which is exactly

$$m(m+1)(m+2)\cdots(m+n) = \frac{(n+m)!}{(m-1)!}$$

as claimed. $\square$

2.3 A convolution identity

We prove one last proposition before applying these techniques to the K-L polynomials.

Definition 2.3.1. Let γ be a complex number and q be a non-negative integer. The *generalized binomial coefficient* of γ and q is

$$\binom{\gamma}{q} = \frac{(\gamma)(\gamma+1)\cdots(\gamma+q-1)}{q!}.$$

Proposition 2.3.2. *Let n, m, and k be integers such that n and $m-k$ are non-negative. Then*

$$\binom{-n}{m-k} = (-1)^{m-k}\binom{n+m-k-1}{m-k}.$$

Proof. By the definition of the generalized binomial coefficient,

$$\binom{-n}{m-k} = \frac{(-n)(-n-1)\cdots(-n-(m-k-1))}{(m-k)!}.$$

We may take -1 out of each of the $m-k$ factors in the numerator to get

$$\binom{-n}{m-k} = (-1)^{m-k}\frac{(n)(n+1)\cdots(n+m-k-1)}{(m-k)!} = (-1)^{m-k}\binom{n+m-k-1}{m-k}$$

as desired. $\square$

The remainder of this thesis will use the more general combinatorial results from this chapter to study the Kuchment-Lvin polynomials defined in the introduction. We begin by expressing the K-L polynomials as linear combinations of differential products of their argument u and providing an alternate proof of the first Kuchment-Lvin identity.

Chapter 3

Combinatorial proof of the first Kuchment-Lvin identity

In all that follows let λ be a complex number, n be a positive integer, and $u = u(x)$ be a smooth function. In this chapter provide an alternate proof of the first Kuchment-Lvin identity:

Theorem (1.0.3)**.** *If u is a smooth function of x and λ is a complex number such that $u' = \lambda u$, then $f_{n,\lambda}(u) = 0$ for all $n \geq 1$.*

3.1 The Kuchment-Lvin polynomials

Recall Definition 1.0.1, reproduced here:

Definition (1.0.1)**.** Let n be a positive integer, λ be a complex number, $u = u(x)$ be a smooth function, and $\partial = \frac{d}{dx}$. The n-th *Kuchment-Lvin (K-L) polynomial* parametrized by λ is defined to be

$$f_{n,\lambda}(u) = u^n + \sum_{k=0}^{n-1} \binom{n}{k} \left(\prod_{m=0}^{n-k-1} (\partial - u + m\lambda) \right) u^k.$$

Example 3.1.1. Compute the $k = 1$ term of $f_{3,\lambda}(u)$,

$$\binom{3}{1}(\partial - u)(\partial - u + \lambda)u.$$

First, we operate on u by $(\partial - u + \lambda)$:

$$(\partial - u + \lambda)u = u' - u^2 + \lambda u.$$

Next, we operate on the result by $(\partial - u)$:

$$(\partial - u)(u' - u^2 + \lambda u) = 3u'' - u'u - 2u'u + u^3 + \lambda u' - \lambda u^2.$$

Finally, we add and multiply by $\binom{3}{1} = 3$.

$$3u'' - 9uu' + 3u^3 + 3\lambda u' - 3\lambda u^2.$$

Example 3.1.2. Take for granted that the $k = 0$ term of $f_{3,\lambda}(u)$ is

$$\binom{3}{0}(\partial - u)(\partial - u + \lambda)(\partial - u + 2\lambda)1$$
$$= -u'' + 3uu' - u^3 - 3\lambda u' + 3\lambda u^2 - 2\lambda^2 u$$

and that the $k = 2$ term is

$$\binom{3}{2}(\partial - u)u^2 = 6uu' - 3u^3.$$

We add these to u^3:

$$f_{3,\lambda}(u) = u^3 + \sum_{k=0}^{2} \binom{3}{k} \left(\prod_{m=0}^{2} (\partial - u + m\lambda) \right) u^k = 2u'' - 2\lambda^2 u.$$

Observe that the coefficients of $f_{3,\lambda}(u)$—by which we mean the integers attached to λ times some product of derivatives of u (see Definition 3.0.32)—sum to zero. In fact, if we assume that $u' = \lambda u$, then $u'' = \lambda^2 u$. Observe that

$$f_{3,\lambda}(u) = \lambda^2 u(2 - 2) = 0,$$

which demonstrates the first and second K-L identities (see Theorems 1.0.3 and 1.0.4, respectively).

Recall from Definition 2.0.6 that $\Pi_{j,\alpha}$ is the set of all differential products

$$\pi = \prod_{m=1}^{j} u^{(\alpha_m)}$$

of derivatives of u with degree j and order $\alpha = \alpha_1 + \cdots + \alpha_j$.

Theorem 3.1.3. *Let n be a positive integer, λ be a complex number, and $f_{n,\lambda}(u)$ be a K-L polynomial. Then*

$$f_{n,\lambda}(u) = \sum_{j=1}^{n} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} \sum_{\pi \in \Pi_{j,\alpha}} C_\pi \pi, \qquad C_\pi \in \mathbf{Z}.$$

Definition 3.1.4. The integer C_π in the above expression is called a *coefficient* of the K-L polynomial.

We prove two lemmas first.

Lemma 3.1.5. *Let k be an integer such that $0 \leq k \leq n-1$. Then*

$$\begin{aligned} &\binom{n}{k} \left(\prod_{m=0}^{n-k-1} (\partial - u + m\lambda) \right) u^k \\ =&\binom{n}{k} \sum_{j=k}^{n} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi} \pi. \end{aligned}$$

Proof. Let j and α be integers such that $0 \leq j-k \leq n-k$ and $0 \leq \alpha \leq n-j$. A term in the expansion of

$$\left(\prod_{m=0} (\partial - u + m\lambda) \right) u^k$$

arises from multiplying by $-u$ a total of $j-k$ times and differentiating α times. The remaining

$$n-k-(j-k)-\alpha = n-j-\alpha$$

operators act as multiplication by $m\lambda$. We add over all possible choices of products of $n-j-\alpha$ integers from $\{1, \ldots, n-k-1\}$. Therefore, there exists a $\beta = (0, \ldots, 0, \beta_k, \ldots, \beta_j) \in Z_{j,\alpha,k}$ (see Notation 2.0.10) such that the result is

$$(-1)^{j-k} \lambda^{n-j-\alpha} S(n-k-1, n-j-\alpha) \partial^{\beta_j} u \partial^{\beta_{j-1}} u \cdots \partial^{\beta_k} u^k$$

where $S(n-k-1, n-j-\alpha)$ is as defined in Definition 2.0.23. After expanding $\partial^{\beta_j} u \partial^{\beta_{j-1}} u \cdots \partial^{\beta_k} u^k$ as in Definition 2.0.11, the above becomes

$$(-1)^{j-k} \lambda^{n-j-\alpha} S(n-k-1, n-j-\alpha) \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi} \pi.$$

Sum over all possible arrangements of ∂ and u—in other words, all $\beta \in Z_{j,\alpha,k}$—and all possible choices of j and α to obtain

$$\sum_{j=k}^{n} (-1)^{j-k} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi} \pi.$$

Multiplying by $\binom{n}{k}$ completes the proof. □

Lemma 3.1.6. *There are no constant terms in the expansion of $f_{n,\lambda}(u)$.*

Proof. Observe that

$$\prod_{m=0}^{n-k-1} (\partial - u + m\lambda)$$

does not decrease the degree of its argument. Therefore, a constant term may only arise from the $k = 0$ term,

$$(\partial - u)(\partial - u + \lambda) \cdots (\partial - u + (n-1)\lambda)1.$$

We seek a term of degree zero. The right-most $n-1$ operators must act as multiplication by $m\lambda$ because multiplication by $-u$ will increase the degree and differentiation will annihilate the constant. Therefore after the first $n-1$ operators we obtain

$$(\partial - u)(n-1)!\lambda^{n-1},$$

which will either be annihilated by ∂ or become $-(n-1)!\lambda^{n-1}u$, a term of degree one. □

Proof of Theorem 3.0.31. Recall that

$$f_{n,\lambda}(u) = u^n + \sum_{k=0}^{n-1} \binom{n}{k} \left(\prod_{m=0}^{n-k-1} (\partial - u + m\lambda) \right) u^k.$$

By Lemma 3.0.33,

$$f_{n,\lambda}(u) = u^n + \sum_{k=0}^{n-1} \binom{n}{k} \sum_{j=k}^{n} (-1)^{j-k} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi}\pi.$$

Trivially,

$$u^n = \binom{n}{n} \sum_{j=n}^{n} \sum_{\alpha=0}^{n-n} \lambda^{n-n-0} S(n-n-1, n-n-0) \sum_{\beta \in Z_{n,0}^n} \sum_{\pi \in \Pi_{n,0}} P_{\beta,\pi}\pi,$$

so we can write

$$f_{n,\lambda}(u) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=k}^{n} (-1)^{j-k} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi}\pi.$$

After re-arranging the sums,

$$f_{n,\lambda}(u) = \sum_{j=0}^{n} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} \sum_{\pi \in \Pi_{j,\alpha}} \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} P_{\beta,\pi}\pi.$$

Define

$$C_\pi = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} P_{\beta,\pi}$$

so that

$$f_{n,\lambda}(u) = \sum_{j=0}^{n} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} \sum_{\pi \in \Pi_{j,\alpha}} C_\pi \pi.$$

By Lemma 3.0.34, there are no terms of degree zero in the expansion of $f_{n,\lambda}(u)$, so we can start the outer sum at $j = 1$. □

3.2 An alternate proof of the first K-L identity

Fix a positive integer n and a complex number λ. We showed in Lemma 2.0.13 that if $u' = \lambda u$ and $\pi \in \Pi_{j,\alpha}$, then $\pi = \lambda^{\alpha} u^{j}$. The K-L polynomial $f_{n,\lambda}(u)$ then becomes

$$f_{n,\lambda}(u) = \sum_{j=1}^{n} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} \sum_{\pi \in \Pi_{j,\alpha}} C_{\pi} \lambda^{\alpha} u^{j} = \sum_{j=1}^{n} \left(\sum_{\alpha=0}^{n-j} \sum_{\pi \in \Pi_{j,\alpha}} C_{\pi} \right) \lambda^{n-j} u^{j}.$$

Notation 3.2.1. Let j be a positive integer and define

$$C_j^{*} = \sum_{\alpha=0}^{n-j} \sum_{\pi \in \Pi_{j,\alpha}} C_{\pi}.$$

Substituting this definition into our expression for $f_{n,\lambda}(u)$ gives

$$f_{n,\lambda}(u) = \sum_{j=1}^{n} C_j^{*} \lambda^{n-j} u^{j}.$$

Clearly if for each $1 \leq j \leq n$ we can prove that $C_j^{*} = 0$, then $f_{n,\lambda}(u) = 0$.

Lemma 3.2.2. *Fix $1 \leq j \leq n$. Let C_j^{*} be as defined in Notation 3.0.35. Then*

$$C_j^{*} = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \sum_{\alpha=0}^{n-j} W(j,\alpha,k) S(n-k-1, n-j-\alpha).$$

Proof. From the proof of Theorem 3.0.31,

$$C_{\pi} = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} P_{\beta,\pi}.$$

Therefore,

$$
\begin{aligned}
C_j^* &= \sum_{\alpha=0}^{n-j} \sum_{\pi \in \Pi_{j,\alpha}} C_\pi \\
&= \sum_{\alpha=0}^{n-j} \sum_{\pi \in \Pi_{j,\alpha}} \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} P_{\beta,\pi}.
\end{aligned}
$$

We re-arrange the sums:

$$
C_j^* = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \sum_{\alpha=0}^{n-j} S(n-k-1, n-j-\alpha) \sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi}.
$$

From the definition of the weight function (see Definition 2.0.19),

$$
\sum_{\beta \in Z_{j,\alpha,k}} \sum_{\pi \in \Pi_{j,\alpha}} P_{\beta,\pi} = W(j, \alpha, k),
$$

so

$$
C_j^* = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \sum_{\alpha=0}^{n-j} W(j, \alpha, k) S(n-k-1, n-j-\alpha). \quad \square
$$

Lemma 3.2.3. *Fix $1 \le j \le n$. Let C_j^* be as defined in Notation 3.0.35. Then*

$$
C_j^* = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \sum_{\alpha=0}^{n-j} W(j, \alpha, k) S(n-k-1, n-j-\alpha) = 0.
$$

Proof. From Lemma 2.0.26,

$$
\sum_{\alpha=0}^{n} m^{\alpha+1} S(n, n-\alpha) = \frac{(m+n)!}{(m-1)!}.
$$

To use Lemma 2.0.26 we must render $S(n-k-1, n-j-\alpha)$ into a form similar to the one that appears in Lemma 2.0.26, namely

$$
S(n-k-1, n-k-1-\alpha).
$$

The goal is achieved by letting γ be such that

$$\alpha = \gamma - j + k + 1.$$

The inner sum over α becomes

$$\sum_{\gamma=j-k-1}^{n-k-1} W(j, \gamma - j + k + 1, k) S(n - k - 1, n - k - 1 - \gamma).$$

If $\gamma < j - k - 1$, then

$$W(j, \gamma - j + k + 1, k) = 0$$

since $\gamma - (j - k - 1) < 0$ (see Definition 2.0.19). We may freely add the zero terms indexed by $0 \leq \gamma \leq j - k - 2$. Our inner sum becomes

$$\sum_{\gamma=0}^{n-k-1} W(j, \gamma - j + k + 1, k) S(n - k - 1, n - k - 1 - \gamma).$$

We now apply Lemma 2.0.22, which says that there exist rational numbers $A_{m,j}$ such that

$$W(j, \gamma - j + k + 1, k) = \sum_{m=k}^{j} \frac{m^{m-k}}{(m-k)!} A_{m,j} m^{\gamma - j + k + 1}.$$

We simplify the above expression:

$$W(j, \gamma - j + k + 1, k) = \sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} m^{\gamma + 1}.$$

We obtain

$$\sum_{\gamma=0}^{n-j} \sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} m^{\gamma+1} S(n - k - 1, n - k - 1 - \gamma)$$
$$= \sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} \sum_{\gamma=0}^{n-j} m^{\gamma+1} S(n - k - 1, n - k - 1 - \gamma).$$

Finally, we use Lemma 2.0.26, which says that

$$\sum_{\gamma=0}^{n-j} m^{\gamma+1} S(n-k-1, n-k-1-\gamma) = \frac{(n-k+m-1)!}{(m-1)!}.$$

This gives us that

$$\begin{aligned}
&\sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} \sum_{\gamma=0}^{n-j} m^{\gamma+1} S(n-k-1, n-k-1-\gamma) \\
&= \sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!}.
\end{aligned}$$

Therefore,

$$C_j^* = \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!}.$$

Now C_j^* is expressed mostly in terms of factorials. It is the right move to fashion binomial coefficients out of these factorials. We re-order the summation:

$$\begin{aligned}
C_j^* &= \sum_{k=0}^{j} (-1)^{j-k} \binom{n}{k} \sum_{m=k}^{j} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!} \\
&= \sum_{k=0}^{j} \sum_{m=k}^{j} (-1)^{j-k} \binom{n}{k} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!} \\
&= \sum_{m=1}^{j} \sum_{k=0}^{m} (-1)^{j-k} \binom{n}{k} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!}
\end{aligned}$$

We multiply and divide by $(n-1)!$:

$$\begin{aligned}
C_j^* &= \sum_{m=1}^{j} \sum_{k=0}^{m} (-1)^{j-k} \binom{n}{k} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!} \\
&= (n-1)! \sum_{m=1}^{j} \sum_{k=0}^{m} (-1)^{j-k} \binom{n}{k} \frac{m^{m-j}}{(m-k)!} A_{m,j} \frac{(n-k+m-1)!}{(m-1)!(n-1)!}
\end{aligned}$$

Next, we multiply and divide by $(-1)^m$:

$$\begin{aligned} C_j^* &= (n-1)! \sum_{m=1}^{j} (-1)^{j-m} \frac{m^{m-j}}{(m-1)!} A_{m,j} \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{k} \frac{(n-k+m-1)!}{(m-k)!(n-1)!} \\ &= (n-1)! \sum_{m=1}^{j} (-1)^{j-m} \frac{m^{m-j}}{(m-1)!} A_{m,j} \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{k} \binom{n-1+m-k}{m-k} \end{aligned}$$

We finally claim that

$$\sum_{k=0}^{m} (-1)^{-k} \binom{n}{k} \binom{n-1+m-k}{m-k} = 0$$

if $m \geq 1$, which would imply that $C_j^* = 0$.

Proposition 2.0.28 says that

$$(-1)^{m-k} \binom{n+m-k-1}{m-k} = \binom{-n}{m-k}$$

so

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{n}{k} \binom{n-1+m-k}{m-k} = \sum_{k=0}^{m} \binom{n}{k} \binom{-n}{m-k}.$$

Using the binomial formula and series convolution, this can be interpreted as the sum of all k-th coefficients of $(1+z)^n$ multiplied by the $m-k$-th coefficient of $(1+z)^{-n}$. Equivalently, this is the m-th coefficient of $(1+z)^n(1+z)^{-n}$, which is the constant function:

$$\sum_{k=0}^{m} \binom{n}{k} \binom{-n}{m-k} = [z^m](1+z)^n(1+z)^{-n} = [z^m]1 = \begin{cases} 0 & m \geq 1 \\ 1 & m = 0 \end{cases}$$

where $[z^m]\phi$ denotes the coefficient on z^m in the series of expansion of some function ϕ.

Of course, the constant function has no nonzero z^m terms if $m \geq 1$. Therefore, $C_j^* = 0$ as claimed. □

Remark 3.2.4. Recall (see Remark 2.0.20) that care was taken to ensure that in the definition of $W(j, \alpha, k)$, the $k = 0$ case reduces to the $k = 1$ case. Therefore, we only consider $m \geq 1$. If $m = 0$, then $(m-1)!$ is undefined, so the claimed value of C_j^* is nonsensical.

Proof of Theorem 1.0.3. Fix a positive integer n and a complex number λ. If $u' = \lambda u$, then using Notation 3.0.35,

$$f_{n,\lambda}(u) = \sum_{j=1}^{n} C_j^* \lambda^{n-j} u^j.$$

Lemma 3.0.37 then gives that $C_j^* = 0$ for all $1 \leq j \leq n$, so $f_{n,\lambda}(u) = 0$. □

3.3 Conclusion

In this chapter, we expressed the polynomial $f_{n,\lambda}(u)$ as a linear combination of differential products of u and used combinatorial identities to give an alternative proof of the first K-L identity. Next, we will apply these techniques to an open questions posed in [4] about the $f_{n,\lambda}(u)$.

Chapter 4

Rejection of new patterns

In the last chapter, we performed combinatorics on the coefficients of the Kuchment-Lvin polynomials to provide an alternate proof of the first K-L identity:

Theorem (1.0.3)**.** *If u is a smooth function and λ is a complex number such that $u' = \lambda u$, then $f_{n,\lambda}(u) = 0$ for all $n \geq 1$.*

We also have the second K-L identity:

Theorem (1.0.4)**.** *If u is a smooth function and λ is a complex number such that $u'' = \lambda^2 u$, then $f_{n,\lambda}(u) = 0$ for all odd n.*

At the end of [4], Kuchment and Lvin asked the following question:

Question (1.0.5)**.** *Does some other pattern occur? Let m be an integer such that $m \geq 3$. If there exists a complex number λ such that $u^{(m)} = \lambda^m u$, then does every m-th K-L polynomial vanish?*

We have found, when we restrict our analysis to the first-degree terms of the K-L polynomial (see Definition 4.0.39 below), that the only patterns that arise are the first and second K-L identities if $\lambda \neq 0$. In Section 4.3, we perform some analysis on patterns that may arise when $\lambda = 0$.

4.1 The linear part of the K-L polynomials

We begin by stating some lemmas necessary to establish the theorem, which largely concern the first-degree terms of $f_{n,\lambda}(u)$. Recall that in our expanded form for the $f_{n,\lambda}(u)$ given in Theorem 3.0.31 that the index j refers to the degree of a term (see Definition 2.0.6).

Definition 4.1.1. The *linear part* of $f_{n,\lambda}(u)$ is the sum of all terms in $f_{n,\lambda}(u)$ where $j = 1$, or

$$f^L_{n,\lambda}(u) = \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} \sum_{\pi \in \Pi_{1,\alpha}} C_\pi \pi.$$

Lemma 4.1.2. *Let $C_\alpha = C_{u^{(\alpha)}}$. Then*

$$f^L_{n,\lambda}(u) = \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C_\alpha u^{(\alpha)}.$$

Proof. We simply need to show that $\Pi_{1,\alpha} = \left\{u^{(\alpha)}\right\}$. This follows from the definition of $\pi \in \Pi_{j,\alpha}$:

$$\pi = \prod_{m=1}^{j} u^{(\alpha_m)} = \prod_{m=1}^{1} u^{(\alpha_m)} = u^{(\alpha_1)} = u^{(\alpha)}$$

since we require $\alpha_1 + \cdots + \alpha_j = \alpha_1 = \alpha$. □

Lemma 4.1.3. *Where the function $S(n, \alpha)$ is defined as in Theorem 2.0.23,*

$$C_\alpha = nS(n-2, n-1-\alpha) - S(n-1, n-1-\alpha).$$

Proof. The C_α are attached to terms in $f_{n,\lambda}(u)$ whose terms have degree $j = 1$. The operator

$$\prod_{m=0}^{n-k-1} (\partial - u + \lambda u)$$

does not decrease the degree of the term. Therefore, we know these terms may only arise from $k = 1$ term,

$$\binom{n}{1}(\partial - u)\cdots(\partial - u + (n-2)\lambda)u,$$

and the $k = 0$ term,

$$\binom{n}{0}(\partial - u)\cdots(\partial - u + (n-1)\lambda)1.$$

Fix α. The term

$$\binom{n}{1}(D-u)\cdots(D-u+(n-2)\lambda)u$$

contributes $u^{(\alpha)}$ when the left-most operator and $\alpha-1$ of the remaining operators act by differentiating u. There are $n-2$ operators other than the left-most one, so $n-2-(\alpha-1)=n-1-\alpha$ of these operators act by multiplication by an integer and λ. (None of these operators will act as multiplication by $-u$; otherwise, we would arrive at a term of degree $j>1$.) Any option gives a product of $n-1-\alpha$ integers between 1 and $n-2$. The sum of all these is

$$S(n-2,n-1-\alpha).$$

We multiply by $\binom{n}{1}=n$ to complete this part of the coefficient.

Likewise, the term

$$\binom{n}{0}(\partial-u)\cdots(\partial-u+(n-1)\lambda)1$$

must act by multiplication by $-u$ exactly once to contribute a term of degree $j=1$. The remaining n operators must contribute α derivatives. This leaves $n-1-\alpha$ operators that act as multiplication by an integer and λ. Adding together all of the options yields

$$S(n-1,n-1-\alpha).$$

Multiplying by $\binom{n}{0}=1$ and adding the contributions of the terms together gives

$$C_\alpha=nS(n-2,n-1-\alpha)-S(n-1,n-1-\alpha)$$

as needed. □

The next lemma provides us with a generating polynomial for the C_α.

Lemma 4.1.4. *Let* $n \geq 2$ *and*

$$h_{n-1}(z) = (n-1)(1-z)\prod_{m=1}^{n-2}(1+mz).$$

Then $h_{n-1}(z)$ *is a generating function for the numbers* $C_{n-1-\alpha}$*, or*

$$h_{n-1}(z) = \sum_{\alpha=0}^{n} C_{n-1-\alpha} z^{\alpha}.$$

Proof. The expression for $h_{n-1}(z)$ looks like the expression for $g_{n-1}(z)$ given in Lemma 2.0.24, which recall is written in summation form as

$$g_{n-1}(z) = \sum_{\alpha=0}^{n-1} S(n-1,\alpha) z^{\alpha}.$$

We will begin with the expression of the C_{α} determined in Lemma 4.0.41,

$$C_{\alpha} = nS(n-2, n-1-\alpha) - S(n-1, n-1-\alpha),$$

and build it into something that allows us to leverage $g_{n-1}(z)$. Replace α with $n-1-\alpha$:

$$C_{n-1-\alpha} = nS(n-2,\alpha) - S(n-1,\alpha).$$

Next, multiply both sides by z^{α} and sum from $\alpha = 0$ to $\alpha = n-1$.

$$C_{n-1-\alpha} z^{\alpha} = nS(n-2,\alpha) z^{\alpha} - S(n-1,\alpha) z^{\alpha}.$$
$$\sum_{\alpha=0}^{n-1} C_{n-1-\alpha} z^{\alpha} = n\sum_{\alpha=0}^{n-1} S(n-2,\alpha) z^{\alpha} - \sum_{\alpha=0}^{n-1} S(n-1,\alpha) z^{\alpha}.$$

Because $S(n-2, n-1) = 0$, we re-index the left sum in the following way:

$$\sum_{\alpha=0}^{n-1} C_{n-1-\alpha} z^{\alpha} = n \sum_{\alpha=0}^{n-2} S(n-2, \alpha) z^{\alpha} - \sum_{\alpha=0}^{n-1} S(n-1, \alpha) z^{\alpha}.$$

By Lemma 2.0.24,

$$h_{n-1}(z) = n g_{n-2}(z) - g_{n-1}(z).$$

Now, we use the product expression for $g_{n-1}(z)$:

$$\begin{aligned} h_{n-1}(z) &= n \prod_{m=1}^{n-2} (1 + mz) - \prod_{m=1}^{n-1} (1 + mz) \\ &= [n - (1 + (n-1)z] \prod_{m=1}^{n-2} (1 + mz) \\ &= (n-1)(1-z) \prod_{m=1}^{n-2} (1 + mz), \end{aligned}$$

which proves the lemma. ☐

Corollary 4.1.5. The roots of $h_{n-1}(z)$ are all rational; particularly, they are $z = 1, -1, -\frac{1}{2}, \ldots, -\frac{1}{n-2}$.

4.2 Proof of the theorem

Definition 4.2.1. Let q be a positive integer. A complex number ζ is called a *q-th root of unity* if $\zeta^q = 1$, and is called *primitive* if $\zeta^n \neq 1$ for all $2 \leq n \leq q-1$.

Theorem 4.2.2. *Let u be a smooth function, λ be a nonzero complex number, and m be an integer such that $m \geq 3$ and $u^{(m)} = \lambda^m u$. Let n be an integer such that $n \geq 2$. If $f^L_{n,\lambda}(u) = 0$, then $u' = \lambda u$ or $u'' = \lambda^2 u$.*

Proof. By the hypothesis that $u^{(m)} = \lambda^m u$, we can solve the characteristic equation

$$y^m = \lambda^m$$

to determine that all solutions to $u^{(m)} = \lambda^m u$ have the form

$$u(x) = \sum_{r=0}^{m-1} \beta_r e^{\lambda \zeta^r x},$$

where ζ is a primitive m-th root of unity and $\beta_0, \ldots, \beta_{m-1}$ are complex scalars.

Recall from Definition 4.0.39 that the linear part of $f_{n,\lambda}(u)$ is given by

$$fn, \lambda^L(u) = \sum_{\alpha=0}^{n-1} C_\alpha u^{(\alpha)}.$$

We will substitute

$$u(x) = \sum_{r=0}^{m-1} \beta_r e^{\lambda \zeta^r x}$$

into $f_{n,\lambda}^L(u) = 0$ and show that the only nonzero scalars are β_0 and, if m is even, $\beta_{\frac{m}{2}}$.

$$\begin{aligned} f^L(u) &= \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C_\alpha \partial^\alpha \left[\sum_{r=0}^{m-1} \beta_r e^{\lambda \zeta^r x} \right] \\ &= \sum_{r=0}^{m-1} \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C_\alpha \beta_r \lambda^\alpha \zeta^{r\alpha} e^{\lambda \zeta^r x} \\ &= \sum_{r=0}^{m-1} \lambda^{n-1} \beta_r \left[\sum_{\alpha=0}^{n-1} \zeta^{r\alpha} C_\alpha \right] e^{\lambda \zeta^r x}. \end{aligned}$$

Recall that the sum

$$\sum_{\alpha=0}^{n-1} \zeta^{r\alpha} C_\alpha = \sum_{\alpha=0}^{n-1} \bar{\zeta}^{-r\alpha} C_\alpha = \bar{\zeta}^{r(1-n)} \sum_{\alpha=0}^{n-1} \bar{\zeta}^{r(n-1-\alpha)} C_\alpha$$

is exactly

$$\bar{\zeta}^{r(1-n)} h_{n-1}(\bar{\zeta}^{-r}) = \zeta^{r(n-1)} h_{n-1}(\zeta^r),$$

where $h_{n-1}(z)$ is the generating polynomial defined in Lemma 4.0.42. Finally, we arrive at

$$\sum_{r=0}^{m-1} \lambda^{n-1} \zeta^{r(n-1)} \beta_r h_{n-1}(\zeta^r) e^{\lambda \zeta^r x} = 0,$$

which is a relation of linear dependence among the distinct functions $e^{\lambda\zeta^r x}$. The $e^{\lambda\zeta^r x}$ are linearly independent, and the only way this relation can hold is if for all r,

$$\lambda^{n-1}\zeta^{r(n-1)}\beta_r h_{n-1}(\zeta^r) = 0.$$

The parameter λ is nonzero, as is ζ, and the only way $h_{n-1}(\zeta^r)$ can be zero is if ζ^r is rational (see Corollary 4.0.43). This implies that if $h_{n-1}(\zeta^r) = 0$, then r must be zero, or if m is even, $\frac{m}{2}$. In either case, we have $h_{n-1}(1) = 0$ or $h_{n-1}(-1) = 0$ and we are forced to conclude that the only nonzero scalars in the expression for u are β_0 and possibly $\beta_{\frac{m}{2}}$ if $\frac{m}{2}$ is an integer. Therefore if m is odd, then

$$u(x) = \beta_0 e^{\lambda x},$$

so $u' = \lambda u$. If m is even, then

$$u(x) = \beta_0 e^{\lambda x} + \beta_{\frac{m}{2}} e^{-\lambda x},$$

so $u'' = \lambda^2 u$. □

4.3 The roots of the linear part

Not only do our methods reveal that $\partial^m u - \lambda^m u$ is not a factor of the decomposition of the linear part of the K-L polynomial, they actually give an entire decomposition. Observe that we can write the linear part in the following way:

$$f^L_{n,\lambda}(u) = \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C^\alpha u^{(\alpha)} = \left(\sum_{\alpha=0}^{n-1} \lambda^{n-1} C_\alpha \partial^\alpha \right)(u).$$

The operator acting on u looks like it may be an instance of the generating function $h_{n-1}(z)$. The next lemma will prove that this is in fact the case.

Lemma 4.3.1. *Let λ be a complex number and n be a positive integer. Let C_α be as found in Lemma 4.0.41. Let u be a smooth function. Then*

$$f_{n,\lambda}(u) = (n-1)(\partial - \lambda)\left(\prod_{a=1}^{n-2}(\partial + a\lambda)\right)(u).$$

Proof. Expand the operator in the right-hand side:

$$\begin{aligned}
&(n-1)(\partial - \lambda)\prod_{a=1}^{n-2}(\partial + a\lambda) \\
&= n\partial\prod_{a=1}^{n-2}(\partial + a\lambda) - (\partial + (n-1)\lambda)\prod_{a=1}^{n-2}(\partial + a\lambda) \\
&= n\partial\prod_{a=1}^{n-2}(\partial + a\lambda) - \prod_{a=1}^{n-1}(\partial + a\lambda).
\end{aligned}$$

For example, expanding

$$\prod_{a=1}^{n-2}(\partial + a\lambda)$$

gives a sum of powers of the operator ∂ times powers of λ times some integer. To obtain ∂^α, we must multiply by $a\lambda$ a total of $n-2-\alpha$ times and add over all possibilities, giving

$$S(n-2, n-2-\alpha)\lambda^{n-2-\alpha}\partial^\alpha.$$

Therefore,

$$\begin{aligned}
&n\partial\prod_{a=1}^{n-2}(\partial + a\lambda) - \prod_{a=1}^{n-1}(\partial + a\lambda) \\
&= n\sum_{\alpha=0}^{n-2} S(n-2, n-2-\alpha)\lambda^{n-2-\alpha}\partial^{\alpha+1} - \sum_{\alpha=0}^{n-2} S(n-2, n-2-\alpha)\lambda^{n-2-\alpha}\partial^\alpha.
\end{aligned}$$

Using the fact that $S(n-2, n-1) = 0$, we re-index:

$$\begin{aligned} & n\sum_{\alpha=0}^{n-1} S(n-1, n-1-\alpha)\lambda^{n-1-\alpha}\partial^{\alpha} - \sum_{\alpha=0}^{n-1} S(n-1, n-1-\alpha)\lambda^{n-1-\alpha}\partial^{\alpha}. \\ &= \sum_{\alpha=0}^{n-1} \left[nS(n-2, n-1-\alpha) - S(n-1, n-1-\alpha)\right]\lambda^{n-1-\alpha}\partial^{\alpha} \\ &= \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C_{\alpha}\partial^{\alpha} \end{aligned}$$

since $C_\alpha = nS(n-2, n-1-\alpha) - S(n-1, n-1-\alpha)$ (see Lemma 4.0.41). □

Theorem 4.3.2. *Let λ be a complex number and n be a positive integer. All roots of $f^L_{n,\lambda}(u)$ are of the form*

$$u(x) = \beta_0 e^{\lambda x} + \sum_{a=1}^{n-2} \beta_a e^{-\lambda a x}$$

where $\beta_0, \beta_1, \ldots, \beta_{n-2}$ are complex.

Proof. Since $f^L_{n,\lambda}(u)$ is a linear differential polynomial, the theorem is equivalent to saying that the solution space of $f^L_{n,\lambda}(u) = 0$ is spanned by

$$\left\{e^{\lambda x}, e^{-\lambda x}, e^{-2\lambda x}, \ldots, e^{-(n-2)\lambda x}\right\}.$$

This is a set of $n-1$ functions and $f^L_{n,\lambda}(u) = 0$ is a $(n-1)$-th order differential equation. Furthermore, the elements in the set are distinct exponential functions and so are linearly independent. Therefore, the set is big enough to span the solution space. It remains to show that all of its members are roots of the linear K-L polynomial.

Let a be arbitrarily chosen from $\{-1, 1, 2, \ldots, n-2\}$. The choice of a corresponds to both an operator $(\partial + a\lambda)$ in the decomposition given in Lemma 4.0.46 and a function $e^{-a\lambda x}$ in the proposed spanning set: in fact,

$$(\partial + a\lambda)e^{-a\lambda x} = -a\lambda e^{-a\lambda x} + a\lambda e^{-a\lambda x} = 0.$$

Observe that since the operators in Lemma 4.0.46 are defined in terms of ∂ and complex scalars, they commute. Therefore, there exists an operator ψ_a such that

$$f^L_{n,\lambda}(e^{-a\lambda x}) = \psi_a \circ (\partial + a\lambda)e^{-a\lambda x} = 0.$$

This gives that all of the members of $\left\{e^{\lambda x}, e^{-\lambda x}, e^{-2\lambda x}, \ldots, e^{-(n-2)\lambda x}\right\}$ are roots of the linear K-L polynomial. Since the set is large enough and is linearly independent, it spans the solution space of $f^L_{n,\lambda}(u) = 0$, and any root of $f^L_{n,\lambda}(u)$ has the form

$$u(x) = \beta_0 e^{\lambda x} + \sum_{a=1}^{n-2} \beta_a e^{-\lambda a x}$$

as claimed. □

4.4 Loosening the $\lambda = 0$ restriction

The reader may wonder, when the original results allow for nonzero λ—and when a feature of the alternate proof given in the third chapter is that we treat $\lambda = 0$ in the same case—why this chapter and the following one require λ to be nonzero.

Notice that key to this proof was writing $u(x)$ as a sum of exponential functions as a consequence of the fact that $u^{(m)} = \lambda^m u$. If $\lambda = 0$, then $u(x)$ is instead a polynomial. Furthermore, if $\lambda = 0$, then the linear part

$$f^L(u) = \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C_\alpha u^{(\alpha)}$$

might vanish without providing us any information about u.

If the index α is less than $n-1$, then that term will be multiplied by zero. Thus all that remains is

$$f^L(u) = C_{n-1} u^{(n-1)} = 0$$

which, again, reveals no information when $n - 1 > m$.

This leads us to state the following theorem, which provides us with information on a potential pattern of $f_{n,0}(u)$ that vanish if $u^{(m)} = 0$.

Theorem 4.4.1. *Let m be a positive integer such that $u^{(m)} = 0$. Suppose there exists a smallest positive integer k such that $f_{k+1,\lambda}(u) = 0$. Then $u^{(k)} = 0$.*

Proof. If $k \geq m$, then successive differentiation yields $u^{(k)} = 0$. The interesting case comes when $k < m$. As previously shown,

$$f^L_{k+1,0}(u) = \sum_{\alpha=0}^{k} \lambda^{k-\alpha} C_\alpha u^{(\alpha)}$$

has disappearing terms for every index except $\alpha = k$. Then

$$f^L_{k+1,0}(u) = C_k u^{(k)} = 0.$$

It remains to show that $C_k \neq 0$. Recall from Lemma 4.0.41 that an expression for C_α is

$$C_\alpha = (k+1)S(k-1, k-\alpha) - S(k, k-\alpha).$$

Then

$$C_k = (k+1)S(k-1, 0) - S(k, 0) = k + 1 - 1 = k.$$

We required $k \geq 1$, so

$$u^{(k)} = 0,$$

which implies that $u^{(k)} = 0$. □

This reveals a relationship between a zero derivative of u with the smallest order and any pattern of $f_{n,0}(u) = 0$ satisfied by u.

Corollary 4.4.2. If m is the smallest integer such that $u^{(m)} = 0$, then any pattern of $f_{n,0}(u)$ that vanish must start at $n \geq m + 1$.

4.5 Conclusion

This chapter revealed the following facts about the roots of the Kuchment-Lvin polynomials:

1. If u is a root of a linear K-L polynomial such that $u^{(m)} = \lambda^m u$ and $\lambda \neq 0$, then either $u' = \lambda u$ or $u'' = \lambda^2 u$.

2. Therefore, the only patterns of vanishing linear K-L polynomials induced by $u^{(m)} = \lambda^m u$ with nonzero λ occur for $m = 1$ and $m = 2$.

3. If $\lambda = 0$, then any pattern of vanishing linear K-L polynomials must begin with the polynomial of index at least $m + 1$.

Certainly there is more to learn about the case where $\lambda = 0$. To obtain more results, we suspect one must move away from the linear part of $f_{n,0}(u)$ to more difficult terms. Also, we have only made claims about the linear part of the polynomial: what can we infer about the whole polynomial? We will return to these problems in the last chapter, where we outline potential directions for future research.

Chapter 5

Rejection of a third Kuchment-Lvin identity

In this chapter, we provide a more concrete proof of the preceding chapter's main result when $m = 3$ using the method of integrating factors.

5.1 Coefficients of the linear polynomial and modular arithmetic

Recall that the C_α are the coefficients of the linear Kuchment-Lvin polynomial (see Definition 4.0.39) with the expression

$$C_\alpha = nS(n-2, n-1-\alpha) - S(n-1, n-1-\alpha)$$

given in Lemma 4.0.41. We will add the C_α according to the congruence class of a α modulo some non-negative integer q. We remind the reader of some facts related to modular arithmetic.

Theorem 5.1.1 (The division algorithm)**.** *Let α and q be integers with α non-negative and q positive. There exist unique integers p and r such that $0 \leq r \leq q-1$ and $\alpha = pq + r$.*

A proof can be found in [20].

Definition 5.1.2. Let α and q be non-negative integers such that $\alpha = pq + r$ for unique integers p and r. We call r the *residue class of α modulo q* and say α is *congruent* to r *modulo* q and write $\alpha \equiv r \bmod q$.

Recall that roots of unity were defined in Definition 4.0.44.

Proposition 5.1.3. *Let $q \geq 2$ be an integer. The sum of all q-th roots of unity is zero.*

Proof. Let ξ be a complex number such that

$$1 + \zeta + \cdots + \zeta^{q-1} = \xi.$$

Multiplying both sides by ζ gives

$$\zeta\xi = \zeta + \zeta^2 + \cdots + \zeta^q = 1 + \zeta + \cdots + \zeta^{q-1} = \xi.$$

Since $q \geq 2$, $\zeta \neq 1$ and therefore $\xi = 0$. □

The primitive roots of unity provide a natural way to divide the C_α according to the residue class of α. If $\alpha = pq + r$, then

$$\zeta^\alpha = \zeta^{pq+r} = (\zeta^q)^p \zeta^r = 1^p \zeta^r = \zeta^r.$$

If γ is a non-negative integer such that both α and γ are congruent to r modulo q, then

$$C_\alpha \zeta^\alpha + C_\gamma \zeta^\gamma = (C_\alpha + C_\gamma)\, \zeta^r.$$

Therefore, a primitive q-th root of unity keeps track of the C_α according to their residue modulo q.

Lemma 5.1.4. *Let $\alpha \equiv r$ mean $\alpha \equiv r \bmod m$ where $m \geq 3$ and put*

$$A_r = \sum_{\alpha \equiv r} C_\alpha.$$

Then there does not exist a number N such that $N = A_r$ for all $0 \leq r \leq m - 1$.

Proof. Let ζ be a primitive m-th root of unity and evaluate $h_{n-1}(z)$ (see Lemma 4.0.42) on its conjugate.

$$\begin{aligned} h_{n-1}(\bar{\zeta}) &= \sum_{\alpha=0}^{n-1} C_\alpha \bar{\zeta}^{n-1-\alpha} \\ &= \zeta^{1-n} \sum_{\alpha=0}^{n-1} C_\alpha \zeta^\alpha, \end{aligned}$$

since $\bar{\zeta} = \zeta^{-1}$.

$$\begin{aligned} h_{n-1}(\bar{\zeta}) &= \zeta^{1-n} \sum_{\alpha=0}^{n-1} C_\alpha \zeta^\alpha \\ &= \zeta^{1-n} \sum_{r=0}^{m-1} \left[\sum_{\alpha \equiv r} C_\alpha \right] \zeta^r. \end{aligned}$$

Suppose that for all r there exists a number N such that $A_r = N$. Then

$$h_{n-1}(\bar{\zeta}) = \zeta^{1-n} N \sum_{r=0}^{m-1} \zeta^r = 0,$$

because by Proposition 5.0.52 all m-th roots of unity sum to zero if $m \geq 2$. However, since $m \geq 3$, it is not the case that $\bar{\zeta}$ is rational or even real. By Lemma 4.0.43, $h_{n-1}(z)$ has only rational roots. Therefore, no such N exists. □

5.2 Rejection of the third identity

Theorem 5.2.1. *Let $\lambda \neq 0$ and let u be a smooth function such that $u^{(3)} = \lambda^3 u$. If $f^L_{n,\lambda}(u) = 0$ for any integer $n \geq 2$, then $u' = \lambda u$.*

Proof. So that we may leverage facts about the second derivative of u, which will only appear in the linear part if $n \geq 3$, we must treat separately the case that $n = 2$. Fix a complex number $\lambda \neq 0$.

If $n = 2$, then

$$\begin{aligned} f_{2,\lambda}(u) &= u + \sum_{k=0}^{1} \binom{2}{k} (\partial - u) \cdots (\partial - u + (1-k)\lambda) u^k \\ &= u^2 + 2(\partial - u)u + (\partial - u)(\partial - u + \lambda)1 \\ &= u^2 + 2u' - 2u^2 - u' + u^2 - \lambda u = u' - \lambda u. \end{aligned}$$

Therefore, if $f_{2,\lambda}(u) = 0$, it must be the case that $u' = \lambda u$.

Let $n \geq 3$. Consider first the case where $u = 0$. In that case, both $u' = \lambda u$ and $u'' = \lambda^2 u$ and the theorem is proven. Instead, suppose $u \neq 0$.

Let u satisfy $u^{(3)} = \lambda^3 u$. Then, in terms of linear operators,

$$u \in \ker \partial^3 - \lambda^3.$$

The operator $\partial^3 - \lambda^3$ decomposes in the following way:

$$\partial^3 - \lambda^3 = (\partial - \lambda)(\partial^2 + \lambda\partial + \lambda^2).$$

Then

$$(\partial^3 - \lambda^3)u = (\partial - \lambda)(\partial^2 + \lambda\partial + \lambda^2)u = (\partial - \lambda)(u'' + \lambda u' + \lambda^2 u) = 0,$$

implying that

$$u'' + \lambda u' + \lambda^2 u \in \ker \partial - \lambda,$$

which in turn tells us that there exists a complex number Γ such that

$$u'' + \lambda u' + \lambda^2 u = \Gamma e^{\lambda x}.$$

We supposed that $f_{n,\lambda}(u) = 0$, which implies that

$$f^L_{n,\lambda}(u) = \sum_{\alpha=0}^{n-1} \lambda^{n-1-\alpha} C_\alpha u^{(\alpha)} = 0.$$

We substitute the fact that $u^{(3)} = \lambda^3 u$ into the linear part of $f_{n,\lambda}(u)$. Let $\alpha \equiv r$ mean $\alpha \equiv r$ mod 3 and put

$$A_r = \sum_{\alpha \equiv r} C_\alpha.$$

Every third derivative of u is $\lambda^3 u$. This means that if there exists an integer p with $\alpha = 3p + r$, then

$$u^{(\alpha)} = \partial^{3p} u^{(r)} = \lambda^{3p} u^{(r)} = \lambda^{\alpha - r} u^{(r)}$$

where $0 \leq r \leq 2$. Therefore,

$$f^L_{n,\lambda}(u) = A_0 \lambda^{n-1} u + A_1 \lambda^{n-2} u' + A_2 \lambda^{n-3} u'' = 0.$$

Next, we use the fact that $u'' + \lambda u' + \lambda^2 u = \Gamma e^{\lambda x}$ to show that

$$f^L_{n,\lambda}(u) = A_0 \lambda^{n-1} u + A_1 \lambda^{n-2} u' + A_2 \lambda^{n-3} \left[\Gamma e^{\lambda x} - \lambda u' - \lambda^2 u \right] = 0,$$

which implies that

$$\lambda^{n-2}[A_2 - A_1] u' + \lambda^{n-1}[A_2 - A_0] u = \Gamma \lambda^{n-3} A_2 e^{\lambda x}.$$

Write

$$B_0 = \lambda^{n-1}[A_2 - A_0], \qquad B_1 = \lambda^{n-2}[A_2 - A_1], \qquad \text{and} \qquad B_2 = \Gamma \lambda^{n-3} A_2.$$

We thus conclude that if $f^L_{n,\lambda}(u) = 0$, then

$$B_1 u' + B_0 u = B_2 e^{\lambda x}.$$

The remainder of the proof will depend on the values of B_0, B_1, B_2, and λ. Recall that we hypothesized that $\lambda \neq 0$ and already addressed the case that $u = 0$. Note also that B_1 and B_0 cannot both be zero. Otherwise, $A_0 = A_1 = A_2$, which contradicts Lemma 5.0.53.

We will consider the cases where B_2 is zero and where it is nonzero and look at the values of B_0 and B_1 as subcases.

Case I. Let $B_2 = 0$. Recall this implies that

$$B_1 u' + B_0 u = 0.$$

Case I.a. Let $B_1 = 0$. In that case, we have that $u = 0$, which has already been addressed.

Case I.b. Let $B_0 = 0$. Then $u' = 0$, implying u is a constant function. However, differentiation gives that

$$u^{(3)} = \lambda^3 u = 0,$$

which implies $\lambda = 0$ since $u \neq 0$. This contradicts the fact that $\lambda \neq 0$, so we cannot have both B_2 and B_0 be zero.

Case I.c. Let neither B_0 nor B_1 be zero. Then since

$$B_1 u' + B_0 u = 0,$$

we conclude that $u' = -\frac{B_0}{B_1} u$. Successive differentiation gives

$$u^{(3)} = -\left(\frac{B_0}{B_1}^3\right) u,$$

which when compared to the fact that $u^{(3)} = \lambda^3 u$, says that $\lambda = -\zeta \frac{B_0}{B_1}$ where ζ is a third root of unity. If $\zeta = 1$, we will have $u' = \lambda u$; we will demonstrate that this must be the case.

We have $B_1 \lambda = -\zeta B_0$. From the definitions of the B_r,

$$\lambda \lambda^{n-2} \left[A_2 - A_1\right] = -\zeta \lambda^{n-1} \left[A_2 - A_0\right].$$

Because $B_0 = \lambda^{n-1}[A_2 - A_0] \neq 0$ we can divide both sides of the equation above by λ^{n-1} and $A_2 - A_0$ to see that

$$-\frac{[A_2 - A_1]}{[A_2 - A_0]} = \zeta.$$

Recall

$$A_r = \sum_{\alpha \equiv r} C_\alpha.$$

The left side is a ratio of sums and differences of integers and is therefore a real number. Then ζ must be one, it must be the case that $\lambda = -\frac{B_0}{B_1}$, and $u' = \lambda u$ as required.

Case II. Let $B_2 \neq 0$. Recall again that

$$B_1 u' + B_0 u = B_2 e^{\lambda x}.$$

Case II.a. Let $B_0 = 0$. Then

$$u' = \frac{B_2}{B_1} e^{\lambda x}$$

and differentiating gives

$$u'' = \lambda \frac{B_2}{B_1} e^{\lambda x}.$$

Applying our knowledge that

$$u'' + \lambda u' + \lambda^2 u = \Gamma e^{\lambda x}$$

gives

$$\lambda \frac{B_2}{B_1} e^{\lambda x} + \lambda \frac{B_2}{B_1} e^{\lambda x} + \lambda^2 u = \Gamma e^{\lambda x}.$$

Since $\lambda \neq 0$, we can solve this equation for u:

$$u = \frac{1}{\lambda^2} \left[\Gamma - 2\lambda \frac{B_2}{B_1} \right] e^{\lambda x},$$

and because u is a constant times the exponential function, we can conclude that $u' = \lambda u$.

Case II.b. Let $B_1 = 0$. Then

$$u = \frac{B_2}{B_0} e^{\lambda x},$$

so $u' = \lambda u$.

Case III.c. Let neither B_1 nor B_0 be zero. Then

$$u' + \frac{B_0}{B_1} u = \frac{B_2}{B_1} e^{\lambda x}.$$

We must divide this case yet again according to the value of λ.

Case III.c.i. Suppose $\lambda = -\frac{B_0}{B_1}$. That would imply that

$$u' - \lambda u = \frac{B_2}{B_1} e^{\lambda x} \neq 0,$$

so we must show that $\lambda \neq -\frac{B_0}{B_1}$. Use the method of integrating factors and multiply both sides of the above by $e^{-\lambda x}$:

$$\begin{aligned} e^{-\lambda x} u' - \lambda e^{-\lambda x} u &= \frac{B_2}{B_1} \\ \frac{d}{dx}\left[e^{-\lambda x} u\right] &= \frac{B_2}{B_1} \\ e^{-\lambda x} u &= \frac{B_2}{B_1} x + K, \qquad K \in \mathbf{C} \\ u &= \frac{B_2}{B_1} x e^{\lambda x} + K e^{\lambda x} \end{aligned}$$

We seek a contradiction. Differentiate u three times and use the assumption $u^{(3)} = \lambda^3 u$:

$$u^{(3)} = \lambda^3 \frac{B_2}{B_1} x e^{\lambda x} + 3\lambda^2 e^{\lambda x} + \lambda^3 K e^{\lambda x} = \lambda^3 \left[\frac{B_2}{B_1} x e^{\lambda x} + K e^{\lambda x}\right].$$

This implies that $3\lambda^2 e^{\lambda x} = 0$, which is false since $\lambda \neq 0$. Therefore, $\lambda \neq -\frac{B_0}{B_1}$. This brings us to the last sub-case.

Case III.c.ii. We will use integrating factors again, now with the knowledge that $\lambda \neq -\frac{B_0}{B_1}$. Recall

$$u' + \frac{B_0}{B_1}u = \frac{B_2}{B_1}e^{\lambda x}.$$

Multiply on both sides by $e^{\frac{B_0}{B_1}x}$:

$$\begin{aligned}
e^{\frac{B_0}{B_1}x}u' + \frac{B_0}{B_1}e^{\frac{B_0}{B_1}x}u &= \frac{B_2}{B_1}e^{(\lambda+\frac{B_0}{B_1})x} \\
\frac{d}{dx}\left[e^{\frac{B_0}{B_1}x}u\right] &= \frac{B_2}{B_1}e^{(\lambda+\frac{B_0}{B_1})x} \\
e^{\frac{B_0}{B_1}x}u &= \frac{\frac{B_2}{B_1}}{\lambda+\frac{B_0}{B_1}}e^{(\lambda+\frac{B_0}{B_1})x} + K, \qquad K \in \mathbf{C} \\
u &= \frac{B_2}{\lambda B_1 + B_0}e^{\lambda x} + Ke^{-\frac{B_0}{B_1}x}
\end{aligned}$$

That $\lambda \neq -\frac{B_0}{B_1}$ guarantees that we are not dividing by zero. Use again the fact that $u^{(3)} = \lambda^3 u$:

$$u^{(3)} = \lambda^3 \frac{B_2}{\lambda B_1 + B_0}e^{\lambda x} + \left(-\frac{B_0}{B_1}\right)^3 Ke^{-\frac{B_0}{B_1}x} = \lambda^3\left[\frac{B_2}{\lambda B_1 + B_0}e^{\lambda x} + Ke^{-\frac{B_0}{B_1}x}\right]$$

only if either $K = 0$ or $\lambda = -\frac{B_0}{B_1}$. Since $\lambda \neq -\frac{B_0}{B_1}$, it must be the case that $K = 0$, and

$$u = \frac{B_2}{\lambda B_1 + B_0}e^{\lambda x}.$$

Therefore $u' = \lambda u$ in all cases that do not otherwise lead to contradictions. $\square$

5.3 Conclusion

In this chapter, we showed that if u is a function such that $u^{(3)} = \lambda^3 u$ where λ is a nonzero complex number, then if u satisfies the linear part of a Kuchment-Lvin differential equation of index $n = 2$ or higher, $u' = \lambda u$.

The proof given here gives a concrete way of learning information about u by using combinatorics and a simple differential equations technique. We have already proven a result (see Theorem 4.0.45) in the cases where $m > 3$ and $u^{(m)} = \lambda^m u$. To give a proof like this one in those cases, the method of integrating factors must be replaced by another technique, possibly the variation of parameters method.

Chapter 6

A note on computation

As Definition 1.0.1 and the expanded form given in Theorem 3.0.31 of the Kuchment-Lvin polynomials may suggest, they become very large—in the sense of number of terms, or space taken on the printed page, or computational complexity—for even small values of their index n. Expanding the definition

$$f_{n,\lambda}(u) = u^n + \sum_{k=0}^{n-1} \binom{n}{k} \left(\prod_{m=0}^{n-k-1} (\partial - u + m\lambda) \right) u^k$$

by hand becomes so difficult around, say, $n = 5$ that one quickly decides to use a computer for this purpose.

According to the end-notes of their original paper [4], a master's student of Kuchment by the name of Ms. E. Rodriguez used the Maple symbolic algebra system to suggest the result of Theorem 5.0.54. We did some of the same work with the Sage mathematics notebook [21], and some discussion is presented here.

6.1 Our Sage code

It is a reasonable assumption that anyone working with the K-L polynomials will at some point need to compute one; therefore, we present and discuss our code here. For obvious reasons, the parameter λ has been renamed in the code to something available on a standard keyboard, particularly a.

We begin with the following observation which allows the $f_{n,\lambda}(u)$ to be generated recursively.

Notation 6.1.1. Let n be a positive integer, λ be a complex number, $u = u(x)$ be a smooth function, and $\partial = \frac{d}{dx}$.

$$\phi_j(u) = \binom{n}{j} u^j + (\partial - u + (n-j)\lambda)\phi_{j-1}(u), \qquad j \geq 1$$
$$\phi_0(u) = 1.$$

Proposition 6.1.2. *Assume the notation from Notation 6.0.55. If $j \geq 1$, then*

$$\phi_j(u) = \binom{n}{j} u^j + \sum_{k=0}^{j-1} \binom{n}{k} \left(\prod_{m=n-j}^{n-k-1} (\partial - u + m\lambda) \right) u^k.$$

Proof. We prove the proposition by induction on j. Let $j = 1$. Then

$$\phi_1(u) = \binom{n}{1} u + (\partial - u + (n-1)\lambda)1.$$

However, we may write $(\partial - u + (n-1)\lambda)1$ as

$$\sum_{k=0}^{0} \binom{n}{0} (\partial - u + (n-1)\lambda) u^0,$$

so

$$\phi_1(u) = \binom{n}{1} u + \sum_{k=0}^{1-1} \binom{n}{k} \left(\prod_{m=n-1}^{n-k-1} (\partial - u + m\lambda) \right) u^k$$

Suppose that the proposition holds for $j-1$. Then

$$\phi_j(u) = \binom{n}{j} u^j + (\partial - u + (n-j)\lambda)\phi_{j-1}(u).$$

By induction,

$$(\partial - u + (n-j)\lambda)\phi_{j-1}(u)$$

becomes

$$\binom{n}{j-1}(\partial - u + (n-j)\lambda)u^{j-1} + \sum_{k=0}^{j-2}\binom{n}{k}\left(\prod_{m=n-1}^{n-k-1}(\partial - u + m\lambda)\right)u^k$$

which can be written as

$$\sum_{k=0}^{j-1}\binom{n}{k}\left(\prod_{m=n-j}^{n-k-1}(\partial - u + m\lambda)\right)u^k.$$

Therefore,

$$\phi_j(u) = \binom{n}{j}u^j + \sum_{k=0}^{j-1}\binom{n}{k}\left(\prod_{m=n-1}^{n-k-1}(\partial - u + m\lambda)\right)u^k$$

as desired. □

Corollary 6.1.3. Let $f_{n,\lambda}(u)$ be the n-th index Kuchment-Lvin polynomial parameterized by λ and evaluated at a smooth function u. Then

$$f_{n,\lambda}(u) = \phi_n(u).$$

Our corollary says that the following code in Figure 6.1 will produce the n-th index K-L polynomial evaluated at u and parametrized by $a = \lambda$. When run, this code can be

```
d = var('d'); u = function('u',x); a = var('a')

def fn(u,a,n):
    gn = 1
    for k in range(1,n+1):
        gn = Combinations(n,k).cardinality() * u^k + diff(gn,x)
             - u*gn + (n-k)*a*gn
    return gn.expand()
```

Figure 6.1: Sage code that outputs $f_{n,a}(u)$.

evaluated either for actual values of u and $\lambda = a$ or the symbolic ones defined on the first line. In Figure 6.2 we present the Sage printout for $n = 8$ to illustrate the size of the K-L polynomials even for small indices. We can adjust our code to check for whether certain

```
-5040*a^7*u(x) + 7308*a^6*u(x)^2 - 2380*a^5*u(x)^3 + 105*a^4*u(
   x)^4 - 7308*a^6*D[0](u)(x) + 3780*a^5*u(x)*D[0](u)(x) +
   3010*a^4*u(x)^2*D[0](u)(x) - 420*a^3*u(x)^3*D[0](u)(x) -
   3325*a^4*D[0](u)(x)^2 + 630*a^2*u(x)^2*D[0](u)(x)^2 + 980*a
   ^5*D[0, 0](u)(x) - 9548*a^4*u(x)*D[0, 0](u)(x) + 3500*a^3*u(
   x)^2*D[0, 0](u)(x) - 630*a^2*D[0](u)(x)^3 - 420*a*u(x)*D[0](
   u)(x)^3 - 1400*a^3*D[0](u)(x)*D[0, 0](u)(x) - 4760*a^2*u(x)*
   D[0](u)(x)*D[0, 0](u)(x) + 6223*a^4*D[0, 0, 0](u)(x) - 7000*
   a^3*u(x)*D[0, 0, 0](u)(x) + 630*a^2*u(x)^2*D[0, 0, 0](u)(x)
   + 105*D[0](u)(x)^4 + 1260*a*D[0](u)(x)^2*D[0, 0](u)(x) +
   3500*a^2*D[0, 0](u)(x)^2 - 1120*a*u(x)*D[0, 0](u)(x)^2 +
   2870*a^2*D[0](u)(x)*D[0, 0, 0](u)(x) - 1260*a*u(x)*D[0](u)(x
   )*D[0, 0, 0](u)(x) + 3920*a^3*D[0, 0, 0, 0](u)(x) - 1708*a
   ^2*u(x)*D[0, 0, 0, 0](u)(x) + 1120*D[0](u)(x)*D[0, 0](u)(x)
   ^2 + 630*D[0](u)(x)^2*D[0, 0, 0](u)(x) + 3500*a*D[0, 0](u)(x
   )*D[0, 0, 0](u)(x) + 1260*a*D[0](u)(x)*D[0, 0, 0, 0](u)(x) +
    1078*a^2*D[0, 0, 0, 0, 0](u)(x) - 140*a*u(x)*D[0, 0, 0, 0,
   0](u)(x) + 315*D[0, 0, 0](u)(x)^2 + 448*D[0, 0](u)(x)*D[0,
   0, 0, 0](u)(x) + 140*D[0](u)(x)*D[0, 0, 0, 0, 0](u)(x) +
   140*a*D[0, 0, 0, 0, 0, 0](u)(x) + 7*D[0, 0, 0, 0, 0, 0, 0](u
   )(x)
```

Figure 6.2: The eighth Kuchment-Lvin polynomial, where the symbol `D[...]u(x)` denotes a derivative of $u(x)$ whose order is equal to the number of zeroes in the brackets.

differential relationships force our K-L polynomials to vanish: The code in Figures 6.3 and 6.4 output a list of which K-L polynomials vanish. It was with this code that we determined the relationship $u^{(3)} = \lambda^3 u$ only caused, out of the first ten polynomials, the first one to vanish (and $f_{1,\lambda}(u) = 0$ identically).

Let us return to the expansion of $f_{8,\lambda}(u)$ given in Figure 6.2. It was looking at exactly this sort of output that suggested Theorem 3.0.31. Note the following things:

```
d = var('d'); u = function('u',x); a = var('a')

def fn(u,a,n):
    gn = 1
    for k in range(1,n+1):
        gn = Combinations(n,k).cardinality() * u^k + diff(gn,x)
            - u*gn + (n-k)*a*gn
        gn = gn.subs_expr(diff(u,x) == a*u)
        % gn = gn.subs_expr(diff(u,x,2) == a^2*u)
        % gn = gn.subs_expr(diff(u,x,3) == a^3*u)
    return gn.expand()
```

Figure 6.3: Sage code that outputs $f_{n,a}(u)$ modulo a relationship like $u' = au$.

```
def scan(u,a,N):
    S = []
    for j in range(1,N+1):
        if fn(u,a,j) == 0:
            S.append(j)
    return S
```

Figure 6.4: Sage code that scans for K-L polynomials that vanish.

1. The degree j of each term—the number of copies of u being multiplied together—does not exceed n. (Indeed, it does not appear to exceed $\lfloor \frac{n}{2} \rfloor$; see the open questions in the next chapter.)

2. The sum α of the orders of the derivatives in each term do not exceed $8 - j$.

3. The power of $\lambda = a$ on each term is $8 - j - \alpha$.

These observations suggest writing $f_{n,\lambda}(u)$ in the form

$$f_{n,\lambda}(u) = \sum_{j=1}^{n} \sum_{\alpha=0}^{n-j} \lambda^{n-j-\alpha} \sum_{\pi \in \Pi_{j,\alpha}} C_\pi \pi,$$

which was proven to be a correct alternate expansion in Theorem 3.0.31. This move was critical to allowing us to use combinatorics to discover facts about the K-L polynomials.

It was stated earlier that only *ten* polynomials were checked to suggest Theorem 5.0.54. The reader may think that ten is a small number. However, even with the simple-looking recursive computation given in the last section, computing the tenth K-L polynomial is a slow process. One potential goal of future research into this area would be discovering a faster way to compute the K-L polynomials—whether by another method, by looking at the way Sage symbolically differentiates functions, or by using another mathematics language.

Chapter 7

Conclusion

In this final chapter, we will summarize the work we did using combinatorics to study the Kuchment-Lvin polynomials. Then, in the spirit of the first paper on this topic, we will close with a list of some possible directions for future research.

7.1 Extending our results to the whole polynomia

We would like to extend Theorem 4.0.45 to the entire K-L polynomial. If u is a smooth function and m is an integer such that $m \geq 3$ and $u^{(m)} = \lambda^m u$, then $u = \sum_{r=0}^{m-1} \beta_r e^{\lambda\zeta^r x}$. Therefore $f_{n,\lambda}(u)$ is a polynomial in the functions $e^{\lambda x}, \ldots, e^{\lambda\zeta^{m-1}x}$.

It is known [22] that if $\gamma_0, \ldots, \gamma_{m-1}$ are linearly independent complex numbers over $\mathbf{Q}$, then $e^{\gamma_0 x}, \ldots, e^{\gamma_{m-1}x}$ are algebraically independent over $\mathbf{C}$, *i.e.* any polynomial with complex coefficients satisfied by $e^{\gamma_0 x}, \ldots, e^{\gamma_{m-1}x}$ must be identically zero.

Therefore, if $\lambda, \lambda\zeta, \ldots, \lambda\zeta^{m-1}$ were linearly independent over $\mathbf{Q}$—which they are not—and $u = \sum_{r=0}^{m-1} \beta_r e^{\lambda\zeta^r x}$, then $f_{n,\lambda}(u)$ must be identically zero when viewed as a polynomial of these exponential functions over $\mathbf{C}$. This would force $f^L_{n,\lambda}(u)$ to be identically zero, which would allow us to extend Theorem 4.0.45 to the whole polynomial.

Of course,

$$\begin{aligned} &\lambda + \lambda\zeta + \cdots + \lambda\zeta^{m-1} \\ &= \lambda\left[1 + \zeta + \cdots + \zeta^{m-1}\right] = 0 \end{aligned}$$

(see Proposition 5.0.52), so something else must be done.

7.2 Closing remarks

In this thesis, we used combinatorics to learn about the K-L polynomials given information about their roots. With numerical evidence provided by the Sage mathematics notebook, we discovered an alternate expansion for a K-L polynomial as a sum of differential products of the argument and proved in Theorem 3.0.31 that this expansion is correct.

Using this expanded form of the polynomial, we proved Theorem 4.0.45, which says that of the differential equations $u^{(m)} = \lambda^m u$, only the equations where $m = 1$ or $m = 2$ are key components in the linear part of the decomposition of the Kuchment-Lvin polynomials.

More broadly, we applied very algebraic and combinatorial methods to what is classically an analytic problem, following the spirit of the new field of differential algebra discussed in Section 1.2.

7.3 Directions for future research

Section 7.1 explained the difficulty in proving Theorem 4.0.45 for the whole polynomial. Here are some other avenues future work on this topic may follow:

- We provided an alternate proof for the first K-L identity (see Theorem 1.0.3) using the expanded form of $f_{n,\lambda}(u)$ given in Theorem 3.0.31. It would be nice to see one for the second Kuchment-Lvin identity (see Theorem 1.0.4) as well.

- Theorem 4.0.48 can be built upon to learn more about the case where $u^{(m)} = 0$ for a positive integer m. This work would most like employ the terms of the K-L polynomial for degree $j > 1$.

- In Section 6.1 we noted that numerical data suggests that the correct upper bound for j in Theorem 3.0.31 is actually $\lfloor \frac{n}{2} \rfloor$, not n. Certainly, there must be a combinatorial reason as to why there are no nonzero terms of degree $j > \lfloor \frac{n}{2} \rfloor$, but we have not discovered it.

- Is there a more efficient method by which to compute the $f_{n,\lambda}(u)$ using Sage or some other mathematics language?

- The K-L polynomials are so intrinsically tied sums of products of integers and the coefficients produced by the product rule (see Chapter 2) that we wonder if, with u and λ chosen correctly, the K-L polynomials do not serve as generating functions for these or related numbers.

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