THE BALANCED VORONOI FORMULAS FOR $\text{GL}(n)$

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Abstract

In this paper we show how the $\text{GL}(N)$ Voronoi summation formula of [MiSc2] can be rewritten to incorporate hyper-Kloosterman sums of various dimensions on both sides. This generalizes a formula for $\text{GL}(4)$ with ordinary Kloosterman sums on both sides that was used in [BLM] to prove nonvanishing of $\text{GL}(4)$ $L$-functions by $\text{GL}(2)$-twists, and later by the second-named author in [Zho].

MSC: 11F30 (Primary), 11F68, 11L05

1 Introduction

The Voronoi summation formula for $\text{GL}(2)$ has long been a standard tool for studying analytic properties of automorphic forms and their $L$-functions. More recently, the Voronoi formula for $\text{GL}(3)$ of the first-named author and Schmid in [MiSc1] has found applications in the study of automorphic forms on $\text{GL}(3)$ and their $L$-functions, such as [Mil], [Li], [Mun], and [FoGa]. The Voronoi formula was generalized to $\text{GL}(N)$ in [MiSc2], with other proofs later found by [GoLi1], [GoLi2], [IcTe], and [KiZh].

The existing Voronoi formula for $\text{GL}(N)$, $N \geq 3$, (e.g., Theorem 2.1) is a Poisson-style summation formula with Fourier coefficients of an automorphic form twisted by additive characters on one side, and those of a contragredient form twisted by (hyper-)Kloosterman sums of dimension $N - 2$ on the other side. The appearance of the (hyper-)Kloosterman sums was already suggested by finite harmonic analysis with Dirichlet characters and Gauss sums, e.g., in [DuIw].

In 2011, the first-named author and Xiaoqing Li discovered a different (so called “balanced”) Voronoi-type formula on $\text{GL}(4)$, with both sides twisted by ordinary Kloosterman sums (see [BLM] and [Zho Theorem 1.2]). This formula was first derived by modifying the automorphic-distributional proof in [MiSc2]. The second-named author later generalized that formula to $\text{GL}(N)$ under certain hypotheses ([Zho Theorem 1.1]). In this paper, we complete the general balanced Voronoi formulas for cusp forms on $\text{GL}(N, \mathbb{Z}) \backslash \text{GL}(N, \mathbb{R})$. These balanced formulas are derived from the original Voronoi formula of [MiSc2], and equate a sum of Fourier coefficients twisted by hyper-Kloosterman sums of dimension $L$ with a contragredient sum twisted by hyper-Kloosterman sums of dimension $M$, where $N = L + M + 2$. The original formula of Miller and Schmid corresponds to the case of $L = 0$ and $M = N - 2$, while the balanced formula of Li and Miller on $\text{GL}(4)$

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corresponds to the case of \( L = 1 \) and \( M = 1 \). The latter formula on \( \text{GL}(4) \) is a key ingredient in the recent nonvanishing theorem for \( \text{GL}(2) \)-twists of \( \text{GL}(4) \) \( L \)-functions in [BLM]. This is because the Kloosterman sums in the balanced Voronoi formula on \( \text{GL}(4) \) mesh well with the Kloosterman sums appearing in the Kuznetsov trace formula on \( \text{GL}(2) \). The match between them is used in [BLM] to create a spectral reciprocity formula, from which mean value estimates and the nonvanishing result are deduced.

The proof of our balanced formulas (Theorem 3.1) in this paper is different from the automorphic-distributional method used to prove Li and Miller’s balanced formula on \( \text{GL}(4) \). Our proof is also different from that of [Zho, Theorem 1.1], which instead uses functional equations of twisted automorphic \( L \)-functions.

Before stating the formulas, we need define the hyper-Kloosterman sums which already appear in the Voronoi formula of [MiSc2] for \( \text{GL}(N) \) for \( N \geq 4 \) (restated in Theorem 2.1 below). Denote \( e(x) := \exp(2\pi ix) \). Let \( a, n \in \mathbb{Z}, c \in \mathbb{N} \), and let

\[
q = (q_1, q_2, \ldots, q_N) \quad \text{and} \quad d = (d_1, d_2, \ldots, d_N)
\]

be \( N \)-tuples of positive integers satisfying the divisibility conditions

\[
d_1 | q_1 c, \quad d_2 | q_1 q_2 c, \quad \ldots, \quad d_N | \frac{q_1 \cdots q_n c}{d_1 \cdots d_{N-1}}.
\]

Define the \( N \)-dimensional hyper-Kloosterman sum as

\[
K_{N}(a, n, c; q, d) := \sum_{\text{x \in } \mathbb{Z}^{\ast}} e \left( \frac{d_1 x_1 a}{c} + \frac{d_2 x_2 x_1}{q_1} \cdot \frac{x_1}{d_1} + \cdots + \frac{d_N x_N x_{N-1}}{q_N} \cdot \frac{x_1}{d_1} \cdot \frac{\cdots x_{N-1}}{d_1} \cdot \frac{c}{d_1} \right),
\]

where \( \sum^{\ast} \) indicates that the summations are restricted to coprime residue classes and \( \tau_i \) denotes the multiplicative inverse of \( x_i \) modulo \( \frac{q_i - q_i c}{d_i} \). In the degenerate case of \( N = 0 \), we define \( K_{0}(a, n, c; \cdot, \cdot) = e \left( \frac{am}{c} \right) \); when \( N = 1 \) the hyper-Kloosterman sum \( K_{1}(a, n, c; q_1, d_1) \) reduces to the ordinary Kloosterman sum \( S(aq_1, n; q_1 c/d_1) \).

Let \( F \) be a cuspidal automorphic form for \( \text{GL}(N, \mathbb{Z}) \). As is customary, we assume that \( F \) generates an irreducible subrepresentation \( \pi \) of \( L_{2}^{2}(\mathbb{Z}_{\mathbb{R}} \text{GL}(N, \mathbb{Z}) \setminus \text{GL}(N, \mathbb{R})) \) under the right regular representation of \( \text{GL}(N, \mathbb{R}) \), where \( \mathbb{Z}_{\mathbb{R}} \) denotes the center of \( \text{GL}(N, \mathbb{R}) \) and \( \xi \) is a central character. Note this does not imply that \( F \) is a Hecke eigenform, which is a stronger assumption that is unnecessary using our methods. Let \( A(\ast, \ldots, \ast) \) denote its abelian Fourier coefficients (see [MiSc2 (2.9)] and [Bum (2.1.5)]), which are the Hecke eigenvalues of \( F \) when \( F \) is a normalized Hecke eigenform. The Voronoi summation formula in [MiSc2] is a Poisson-sum style identity relating sums of the abelian Fourier coefficients weighted against test functions \( \omega \) and \( \Omega \), which are related by an integral transform completely determined by \( \pi \). Further background on Voronoi summation and this integral transform (which our new formula shares as well) is given in Section 2.

There are various ways to describe allowable choices of test functions \( \omega \) in the Voronoi summation formula. The simplest approach (which we follow here) is to demand that \( \omega \) be a smooth function on \( \mathbb{R} \) which has compact support contained in \( \mathbb{R}_{>0} = (0, \infty) \); this is natural since \( \omega(x) \) is never evaluated at \( x = 0 \) in Theorem 1.1. However, for some applications (e.g., to \( L \)-functions) it
is important to allow different behavior at the origin, such as fractional powers of the form \(|x|^s\) or \(|x|^s \text{sgn}(x)|\) for \(s \in \mathbb{C}\). We shall not pursue this here, other than noting that any admissible function used in the usual Voronoi formula on \(GL(N)\) (see [MiSc2, (1.8)]) can be used in the balanced Voronoi formulas (with only minor modifications to account for parities); this is because our proof constructs the balanced formula as a finite average of formulas of the type given in Theorem 2.1.  

At a formal level, the integral transform has the form

\[
\Omega(y) = \frac{1}{|y|} \int_{\mathbb{R}^N} \omega \left( \frac{x_1 \cdots x_N}{y} \right) \prod_{1 \leq j \leq N} \left( e(-x_j) |x_j|^{-\lambda_j} \text{sgn}(x_j)^{\delta_j} \, dx_j \right), \quad (2)
\]

where the \(\lambda_j\) and \(\delta_j\) are the representation parameters of \(\pi\) (this notion as well as a reformulation of (2) in terms of Mellin inversion is given in Section 2; see also [MiSc2, §I]).

**Theorem 1.1.** Let \(F\) be a cuspidal automorphic form on \(GL(N,\mathbb{Z}) \setminus GL(N,\mathbb{R})\) for \(N \geq 3\) with abelian Fourier coefficients \(A(*,\ldots,\cdot)\), and which generates an irreducible representation of \(GL(N,\mathbb{R})\). Let \(\omega \in C_c^\infty(\mathbb{R}_{>0})\) and let \(L\) and \(M\) be two non-negative integers with \(L+M+2 = N\). Let \(c > 0\) be an integer and let \(a\) be an integer with \((a,c) = 1\). Denote by \(\bar{a}\) the multiplicative inverse of \(a\) modulo \(c\). Let \(\mathbf{q} = (q_1, q_2, \ldots, q_L)\) be an \(L\)-tuple of positive integers and let \(\mathbf{Q} = (Q_1, Q_2, \ldots, Q_M)\) be an \(M\)-tuple of positive integers. Let \(\sum d_{|d|}^{D(Q)} \prod_{d|q} \omega(nD_1^{D_1}D_2^{D_2} \cdots D_M^{D_M})\) stand for \(\sum_{d_1|q_1} \sum_{d_2|q_2} \sum_{d_L|q_L} \omega(nD_1^{D_1}D_2^{D_2} \cdots D_M^{D_M})\) and let

\[
\sum_{d|q} \sum_{\mathbf{D(Q)}} A(q_1, \ldots, q_L, D_1, \ldots, D_M, n) \text{Kl}_M(\bar{a}, n, c; \mathbf{Q}, \mathbf{D}) D_1^{D_1} D_2^{D_2} \cdots D_M^{D_M} \omega(nD_1^{D_1}D_2^{D_2} \cdots D_M^{D_M})
\]

\[
= \sum_{d|q} d_1^{L-1} \cdots d_L \sum_{n=1}^\infty A(n, d_1, \ldots, q_L) \text{Kl}_L(n, -\mathbf{Q}; \mathbf{d}) \omega(nD_1^{D_1}D_2^{D_2} \cdots D_M^{D_M})
\]

\[
+ \sum_{d|q} d_1^{L-1} \cdots d_L \sum_{n=1}^\infty A(n, d_1, \ldots, q_L) \text{Kl}_L(n, -\mathbf{Q}; \mathbf{d}) \omega(nD_1^{D_1}D_2^{D_2} \cdots D_M^{D_M})
\]

where \(\Omega\) is the integral transform from (2) (which is rigorously defined as a convergent integral in [MiSc2, §II]).

**Remark 1.2.** As we mentioned earlier, Theorem 1.1 is proved by averaging over a finite number of instances of the original Voronoi formula of [MiSc2] (Theorem 2.1). Consequently, any analysis of test functions for that formula automatically transfers over to our present setting. The construction by finite average also shows that any coefficients \(A(*,\ldots,\cdot)\) satisfying the summation formula in [MiSc2] must also satisfy the summation formula in Theorem 3.1. This extends the range of applicability of \(F\) to cases where functoriality has not yet been shown. For example, Kiral and the second-named author have shown in [KiZh] that the Voronoi summation formula of [MiSc2] also holds when \(F\) is a Rankin-Selberg convolution of two full-level cuspidal automorphic representations (see [KiZh] Examples 1.8-1.9]). Therefore Theorem 1.1 and Theorem 3.1 hold for such \(F\) as well, despite it not yet being known to be automorphic.

**Remark 1.3.** We have stated the summation formula in Theorem 1.1 so that it only involves a sum over positive integers \(n\) on the lefthand side. This is somewhat unnatural from the point of view of automorphic distributions, through which one obtains summation formulas via integration.
against distributions that involve terms for both positive and negative \(n\). Also, including both positive and negative \(n\) on the lefthand side results in simplifying the righthand side, as well as the analytic assumptions on the behavior of \(\omega\) near the origin. Nevertheless, since Voronoi summation formulas are typically applied to sums indexed by positive integers \(n\), we have chosen to sacrifice aesthetics for practicality and state our formula as above.

2 Voronoi formulas as Dirichlet series identities

Let \(F\) be a cuspidal automorphic form on \(GL(N, \mathbb{Z}) \backslash GL(N, \mathbb{R})\) and let \(\pi\) denote the archimedean representation attached to \(F\), which we assume is irreducible. We say that \((\lambda, \delta) \in \mathbb{C}^N \times (\mathbb{Z}/2\mathbb{Z})^N\) is a representation parameter of \(F\) if \(\pi\) embeds into a subspace of the principal series representation

\[
V_{\lambda, \delta} = \left\{ f : GL(N, \mathbb{R}) \to \mathbb{C} \left| f \left( g \left( \begin{array}{ccc} a_1 & 0 & 0 \\ \ast & \ddots & 0 \\ \ast & \cdots & a_n \end{array} \right) \right) = f(g) \prod_{j \leq N} (|a_j|^{N+1/2 - j - \lambda_j} \text{sgn}(a_j)^{\delta_j}) \right\},
\]

which is a representation space for \(GL(N, \mathbb{R})\) under the left translation action \([\pi_{\lambda, \delta}(g)f]h = f(g^{-1}h)\).

When \(\pi\) is spherical, any simultaneous permutation of the entries of \((\lambda, \delta)\) is also a representation parameter; in this case \(\lambda\) coincides with the notion of Langlands parameter, though it does not in general (see [MSc3] A.1-A.2 for a complete description of all allowable representation parameters of cuspidal automorphic representations of \(GL(N, \mathbb{R})\)). The \(GL(N, \mathbb{Z})\)-invariance forces \(\delta_1 + \cdots + \delta_n \equiv 0 \pmod{2}\) [MSc2] (2.2).

Define the Gamma factor

\[
G_{\delta}(s) := \begin{cases} 2(2\pi)^{-s/2} \Gamma(s) \cos(\pi s/2), & \text{if } \delta \in 2\mathbb{Z}, \\ 2i(2\pi)^{-s/2} \Gamma(s) \sin(\pi s/2), & \text{if } \delta \in 2\mathbb{Z} + 1. \end{cases}
\]

Alternatively, if \(\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)\) denotes the usual Artin Gamma factor appearing in the functional equation of the Riemann \(\zeta\)-function, then we have equivalently

\[
G_{\delta}(s) = \begin{cases} \frac{\Gamma_R(s)}{\Gamma_R(1-s)}, & \delta \in 2\mathbb{Z}, \\ \frac{i \Gamma_R(s+1)}{\Gamma_R(2-s)}, & \delta \in 2\mathbb{Z} + 1. \end{cases}
\] (3)

Define

\[
G_+(s) = \prod_{j=1}^{N} G_{\delta_j}(s + \lambda_j)^{-1} \quad \text{and} \quad G_-(s) = \prod_{j=1}^{N} G_{1+\delta_j}(s + \lambda_j)^{-1}.
\] (4)

The ratio of Gamma factors \(G_{\pm}(s)\) appears naturally in the functional equations of the standard \(L\)-function of \(F\) and its twists by Dirichlet characters. For this reason we can alternatively write

\[
G_+(s) = \frac{L(1 - s, \overline{\pi}) \varepsilon(s, \pi)}{L(s, \pi)} \quad \text{and} \quad G_-(s) = \frac{L(1 - s, \overline{\pi} \otimes \text{sgn}) \varepsilon(s, \pi \otimes \text{sgn})}{L(s, \pi \otimes \text{sgn})},
\] (5)

where the local factors are as defined in [Jac] Appendix.

Let \(\omega \in C_c^\infty(\mathbb{R}_{>0})\) and let \(\hat{\omega}(s)\) denote its Mellin transform. We shall now clarify the relationship between \(\omega\) and its Voronoi transform \(\Omega\) from [2]. Decompose \(\Omega\) into its even and odd parts for \(y > 0\)

\[
\Omega_+(y) = \frac{1}{2} (\Omega(y) + \Omega(-y)) \\
\Omega_-(y) = \frac{1}{2} (\Omega(y) - \Omega(-y)).
\] (6)
It then follows from [MiSc2, (1.5)] that
\[ \Omega_{\pm}(x) = \frac{1}{2\pi i} \int_{\Re(s) = -\sigma} \tilde{\omega}(s) x^{s-1} G_{\pm}(s) \, ds \quad (7) \]
for \( x > 0 \) and some \( \sigma > 0 \). Please note that we take some \( \sigma > 0 \) to avoid the poles of \( G_{\pm}(s) \), which are on some right half plane. Also, \( \Omega \) is defined over \( \mathbb{R} \setminus \{0\} \) and \( \Omega_{\pm} \) over \( \mathbb{R}_{>0} \).

The original Voronoi formula for \( \text{GL}(N) \), \( N \geq 3 \), in [MiSc1, MiSc2] was proven using automorphic distributions. The methods of [MiSc2, §4] can be used to derive Theorem 1.1 as well. We shall however prove it using a reformulation in terms of Dirichlet series, which we state in Theorem 3.1. This reformulation will itself be proved by taking a finite average of a similarly-reformulated version of the GL(\( N \)) Voronoi summation formula in terms of Dirichlet series, which can be found in [KiZh] (and [MiSc2] (1.12)):

**Theorem 2.1** (Voronoi formula on \( \text{GL}(N) \) of Miller-Schmid [MiSc2]). Let \( F \) be a cuspidal automorphic form on \( \text{GL}(N, \mathbb{Z}) \backslash \text{GL}(N, \mathbb{R}) \) with abelian Fourier coefficients \( A(*, \ldots, *) \). Assume that \( F \) generates an irreducible representation \( \pi \) of \( \text{GL}(N, \mathbb{R}) \) and let \( G_{\pm} \) be the ratio of Gamma factors from (4)-5. Let \( c > 0 \) be an integer and let \( a \) be any integer with \( (a, c) = 1 \). Denote by \( \overline{a} \) the multiplicative inverse of \( a \) modulo \( c \). Let \( q = (q_1, q_2, \ldots, q_{N-2}) \) be an \( (N-2) \)-tuple of positive integers. Then the additively-twisted Dirichlet series
\[ L_q(s, F, \overline{a}/c) = q_1^{(N-2)s} q_2^{(N-3)s} \cdots q_{N-2}^{s} \sum_{n=1}^{\infty} \frac{A(q_{N-2}, \ldots, q_1, n)}{n^s} e\left( \frac{\overline{a}n}{c} \right) \quad (8) \]
which is initially convergent for \( \Re s \gg 1 \), has an analytic continuation to an entire function of \( s \in \mathbb{C} \) satisfying the functional equation
\[ L_q(s, F, \overline{a}/c) = \]
\[ \frac{G_+(s) - G_-(s)}{2} \sum_{d_1 | q_1 c} \sum_{d_2 | q_2 c} \cdots \sum_{d_{N-2} | q_{N-2} c} \sum_{n=1}^{\infty} \frac{A(n, d_{N-2}, \ldots, d_2, d_1) Kl_{N-2}(a, n, c; q, d)}{n^{1-s} c^{N-1} d_1^{1-(N-1)s} d_2^{1-2s} \cdots d_{N-2}^{1-2s}} \]
\[ + \frac{G_+(s) + G_-(s)}{2} \sum_{d_1 | q_1 c} \sum_{d_2 | q_2 c} \cdots \sum_{d_{N-2} | q_{N-2} c} \sum_{n=1}^{\infty} \frac{A(n, d_{N-2}, \ldots, d_2, d_1) Kl_{N-2}(a, -n, c; q, d)}{n^{1-s} c^{N-1} d_1^{1-(N-1)s} d_2^{1-2s} \cdots d_{N-2}^{1-2s}} \quad (9) \]
where \( d = (d_1, \ldots, d_{N-2}) \) (both terms on the righthand side converge for \( \Re s \ll -1 \) and have entire continuations to \( s \in \mathbb{C} \)).

**3 Proof**

We begin by restating Theorem 1.1 in the language of Dirichlet series, analogously to Theorem 2.1.

**Theorem 3.1.** Let \( F \) be a cuspidal automorphic form on \( \text{GL}(N, \mathbb{Z}) \backslash \text{GL}(N, \mathbb{R}) \), \( N \geq 3 \), with abelian Fourier coefficients \( A(*, \ldots, *) \). Assume that \( F \) generates an irreducible representation \( \pi \) of \( \text{GL}(N, \mathbb{R}) \) and let \( G_{\pm} \) be the ratio of Gamma factors from (4)-5. Let \( L \) and \( M \) be two non-negative integers whose sum \( L + M = N - 2 \). Let \( c > 0 \) be an integer and let \( a \) be any integer with \[ The formula stated here corrects a misprint propagating from MiSc2 (1.9), where the first two arguments in the definition of the Kloosterman sum were mistakenly switched.
$(a, c) = 1$. Denote by $\bar{a}$ the multiplicative inverse of $a$ modulo $c$. Let $q = (q_1, q_2, \ldots, q_L)$ be an $L$-tuple of positive integers and $Q = (Q_1, Q_2, \ldots, Q_M)$ an $M$-tuple of positive integers. Define the Dirichlet series

$$L_{q, Q}(s, F, \bar{a}/c) = \sum_{D \mid Q} \prod_{n=1}^{\infty} A(q_L, \ldots, q_1, D_1, \ldots, D_M, n) \text{Kl}_M(\bar{a}, n, c; Q, D) \frac{q_1^{L_1} \cdots q_L^{L_L}}{D_1^{(M+1)s-M} \cdots D_M^{2s-1}}$$

(10)

where $\sum_{D \mid Q}$ stands for $\sum_{D_1 \mid Q_1} \sum_{D_2 \mid Q_2} \cdots \sum_{D_M \mid Q_M}$. This Dirichlet series is convergent for $\Re s \gg 1$, and has an analytic continuation to an entire function in $s \in \mathbb{C}$ which satisfies the functional equation

$$L_{q, Q}(s, F, \bar{a}/c) = c^{M+1-Ns} \left[ G_+(s) + (-1)^{M+1} G_-(s) \right] L_{q, Q}(1-s, \bar{F}, a/c) + \frac{G_+(s) + (-1)^M G_-(s)}{2} L_{q, Q}(1-s, \bar{F}, -a/c)$$

(11)

where $L_{q, Q}(\ldots, \bar{F}, \ldots)$ is defined using the contragredient coefficients $\bar{A}(m_1, \ldots, m_{n-1}) = A(m_{n-1}, \ldots, m_1)$.

The following two lemmas are used in the proof of Theorem 3.1.

**Lemma 3.2.** Let $C, Q$, and $b$ be positive integers and $y, a$ integers. Assuming $b \mid QC$, $(y, QC/b) = 1$ and $(a, C) = 1$, we have

$$\sum_{D \mid QC} \sum_{x \pmod{QC}} e\left(\frac{Dxa}{C} + \frac{byx}{QC}\right) = \begin{cases} QC, & \text{if } b = Q \text{ and } y \equiv -a \pmod{C}, \\ 0, & \text{otherwise.} \end{cases}$$

(12)

**Proof.** The sum $\sum_{z \pmod{QC}} e\left(\frac{z(Qa+by)}{QC}\right)$ equals $QC$ when $QC \mid Qa + by$, and vanishes otherwise. Factoring each $z$ as $z = Dx$ with $D = \gcd(z, QC)$ and $x \in (\mathbb{Z}/QC\mathbb{Z})^*$, we see this sum equals the lefthand side of (12). It thus suffices to show that the nonvanishing conditions are equivalent. Clearly $QC \mid Qa + by$ if $b = Q$ and $y \equiv -a \pmod{C}$. Conversely, suppose $Q \mid Qa + by$. Thus $Q \mid by$, which implies that $Q$ divides $\gcd(by, QC) = b$; also, since we have assumed that $b \mid QC$, we must have $b \mid Qa$ and hence $b$ divides $\gcd(Qa, QC) = Q$. Being divisors of each other, $b$ and $Q$ are equal; this forces $C \mid (a + y)$.

**Proof of Theorem 3.1.** We open up the hyper-Kloosterman sums on the lefthand side of (10) com-
By Theorem 2.1, the \( n \)-sum part is absolutely convergent for \( R s \gg 1 \) and has analytic continuation to \( C \), hence the same assertions are true of (10). Applying (9) to the \( n \)-sum, we get

\[
\sum_{\mathbf{D} \mid \mathbf{Q}} D_1^{M-1} D_2^{M-2} \cdots D_M^{M-1} \sum_{x_1 \pmod{Q_1 D_1}} \cdots \sum_{x_M \pmod{Q_M D_M}} e\left( \frac{D_1 x_1 \bar{a} + D_2 x_2 \bar{a_T} + \cdots + D_M x_M \bar{a_{M-1}}}{c} + \frac{n x_M}{Q_1 \cdots Q_M D_1 \cdots D_M} \right)
\]

which is absolutely convergent for \( R s \ll -1 \). We open up the hyper-Kloosterman sum partially, obtaining

\[
\text{Kl}_{N-2} \left( x_M, n, \frac{Q_1 \cdots Q_M c}{D_1 \cdots D_M}; (D_M, \cdots, D_1, q_1, \cdots, q_L), (b_M, \cdots, b_1, d_1, \cdots, d_L) \right)
\]

\[
= \sum_{y_M \pmod{b_M D_M}} y_M \sum_{y_{M-1} \pmod{b_M D_M^{M-1}}} \cdots \sum_{y_1 \pmod{b_M D_M^{M-1}}} e\left( \frac{b_M y_M x_M}{Q_1 \cdots Q_M D_1 \cdots D_M} + \frac{b_M-1 y_{M-1} y_M}{Q_1 \cdots Q_M D_1 \cdots D_M} + \cdots + \frac{b_1 y_1 \bar{y}_2}{Q_1 \cdots Q_M D_1 \cdots D_M} \right) \text{Kl}_L \left( \bar{y}_1, n, \frac{Q_1 \cdots Q_M c}{b_1 \cdots b_M}; (q_1, \cdots, q_L), (d_1, \cdots, d_L) \right).
\]
After reordering the summations, $L_{a/c}Q(s, F, \bar{a}/c)$ equals

\[
\sum_{D_1 | Q_1c} \sum_{x_1 \pmod{\frac{Q_1c}{D_1}}} \cdots \sum_{D_{M-1} | Q_{M-1}c} \sum_{x_{M-1} \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} \sum_{D_M | \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}} \sum_{x_M \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} (14a)
\]

\[
\sum_{b_M \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}c}}} b_M \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}c} \sum_{y_M \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}c}}} y_M \cdots \sum_{b_1 \pmod{\frac{Q_1c}{D_1}}} b_1 \frac{Q_1c}{D_1} \sum_{y_1 \pmod{\frac{Q_1c}{D_1}}} y_1 (14b)
\]

\[
\sum_{d_1 | q_1} \sum_{d_2 | q_2} \cdots \sum_{d_L | q_L} e \left( \frac{D_1 x_1 \bar{a}}{c} + \frac{D_2 x_2 \bar{a}}{c} + \cdots + \frac{D_M x_M \bar{a}}{c} - \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}c} \sum_{x_{M-1} \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} + \frac{b_M y_M x_M}{D_M} + \frac{b_{M-1} y_{M-1} x_{M-1}}{D_{M-1}} + \cdots + \frac{b_1 y_1 x_1}{D_1} \right) (14c)
\]

\[
\sum_{n=1}^{\infty} A(n, d_L, \ldots, d_1, b_1, \ldots, b_M) \left[ \frac{\zeta(s) - \zeta(s)}{2} \right] K_L \left( \frac{y_1, n, Q_{M-1}c}{b_1 \cdots b_M} ; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right) + \frac{\zeta(s) + \zeta(s)}{2} K_L \left( \frac{y_1, -n, Q_{M-1}c}{b_1 \cdots b_M} ; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right)
\]

Observe that $D_M$ is not present in the summations in lines (14b) and (14c). Thus consider the $D_{M-1}$- and $x_{M-1}$-summations,

\[
\sum_{D_{M-1} | \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}} \sum_{x_{M-1} \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} e \left( \frac{D_{M-1} x_{M-1} \bar{x}_{M-1}}{c} + \frac{b_{M-1} y_{M-1} x_{M-1}}{D_M} \right)
\]

\[
\sum_{D_{M-1} | \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}} \sum_{x_{M-1} \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} \sum_{D_M | \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}} \sum_{x_M \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} e \left( \frac{D_M x_M \bar{x}_M}{c} - \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}c} \sum_{x_{M-1} \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} - \frac{b_M y_M x_M}{D_M} \right)
\]

\[
\sum_{D_{M-1} | \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}} \sum_{x_{M-1} \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} \sum_{D_M | \frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}} \sum_{x_M \pmod{\frac{Q_{M-1}c}{D_1 \cdots D_{M-1}}}} e \left( \frac{D_{M-1} x_{M-1} \bar{x}_{M-1}}{c} + \frac{b_{M-1} y_{M-1} x_{M-1}}{D_M} \right)
\]

This forces $b_{M-1} = Q_{M-1}$ and $y_{M-1} \equiv (-1)^2 \bar{x}_{M-2} \pmod{\frac{Q_{M-2}c}{D_1 \cdots D_{M-2}}}$.
forcing $b_j = Q_j$ and $y_j \equiv (-1)^{M-j+1} \overline{x_j^{-1}} \pmod{\frac{Q_1 \cdots Q_j - 1}{D_1 \cdots D_{j-1}}}$. At the final stage, we apply Lemma 3.2 once more to the remaining sum

$$
\sum_{D_1 | Q_1 c} \sum_{x_1 (\text{mod} \ \frac{Q_1 c}{D_1})} \sum^* e \left( \frac{D_1 x_1 \bar{a}}{c} + \frac{(-1)^M b_1 y_1 x_1}{Q_1 c} \right)
$$

to force $b_1 = Q_1$ and $y_1 \equiv (-1)^M \bar{a} \pmod{c}$. At this point the summations in (14a) and (14b) have all disappeared, and we find (13) is equal to

$$
\sum_{d_1 | q_1 c} \cdots \sum_{d_{L+1} | \prod_{i=1}^{L+1} D_i} \sum_{n=1}^{\infty} \frac{A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) c^M Q_1^M \cdots Q_M}{\prod_{i=1}^{L+1} d_i^{1-2s}}
$$

$$
\times \left[ \frac{G_+(s) - G_-(s)}{2} K_{L_1} \left( (-1)^M a, n, c; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right) \right.
$$

$$
+ \left. \frac{G_+(s) + G_-(s)}{2} K_{L_1} \left( (-1)^M a, -n, c; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right) \right]
$$

$$
= \frac{G_+(s) + (-1)^{M+1} G_-(s)}{2} \sum_{d | q} \sum_{n=1}^{\infty} \frac{A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) K_{L_1}(a, n, c; q, d)}{n^{1-s} c^{N-1-M}} \frac{Q_1^{M(1-s)} \cdots Q_M^{1-s}}{d_1^{1-2s}}
$$

$$
+ \frac{G_+(s) + (-1)^M G_-(s)}{2} \sum_{d | q} \sum_{n=1}^{\infty} \frac{A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) K_{L_1}(a, -n, c; q, d)}{n^{1-s} c^{N-1-M}} \frac{Q_1^{M(1-s)} \cdots Q_M^{1-s}}{d_1^{1-2s}},
$$

which is equivalent to (11). □

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