Course website

http://math.umaine.edu/~ormerod/mat126/
What did we do last lecture?

2nd of September 2016
One sided limits and two sided limits

If we have a function, $f(x)$, in which

$$
\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x),
$$

then we say that

$$
\lim_{x \to a} f(x)
$$

does not exist. Otherwise we have

$$
\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).
$$
One sided limits and two sided limits

Given a function whose graph is the following:

When is the limit not defined? Why?
Continuous functions

A function is said to be continuous at \( x = a \) if

- \( f(a) \) exists.
- \( \lim_{x \to a} f(x) \) exists.
- \( \lim_{x \to a} f(x) = f(a) \).

The function \( f \) is said to be continuous if it is continuous at each point in the function's domain.

This formalizes what we mean by “nice”.
Discontinuities

We different types of discontinuity:

- Removable: all thats missing is a value where a limit exists

\[ f(x) = \frac{x^2 - 3x + 2}{x - 1} \]

- A jump: pen off the page to draw.

\[ f(x) = \frac{\sin(x)}{|x|} \]
Discontinuities

We have three types of discontinuity

▶ Asymptote: the function goes off to $\infty$

▶ There is one more that is odd.
Is the following continuous

Let \( f(x) \) be the function given by the following graph

Is this function continuous? If not, why?
Functions without limits

The last type of discontinuity is where there simply is no limit. Take

\[ f(x) = \sin \left( \frac{1}{x} \right), \]

around \( x = 0 \).
Functions without limits

The last type of discontinuity is where there simply is no limit. Take

$$f(x) = \sin \left( \frac{1}{x} \right),$$

around \( x = 0. \)
Functions without limits

The last type of discontinuity is where there simply is no limit. Take

\[ f(x) = \sin \left( \frac{1}{x} \right), \]

around \( x = 0 \).

How do you show the limit is not defined?
Functions without limits

Let us consider the function

\[ f(x) = \sin \left( \frac{1}{x} \right), \]

in more detail on a more limited set.
Continuous functions

The simplest example of a continuous function is a line

\[ f(x) = ax + b. \]

So for example, if

\[ \lim_{x \to 5} 2x + 3 = \]

Limit laws

Assume that both these exist,

$$\lim_{x \to a} f(x), \quad \lim_{x \to a} g(x),$$

then the following hold:

1. The limit of $f(x) + g(x)$ at $x = a$ exists and

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. The limit of $f(x) - g(x)$ at $x = a$ exists and

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$
Limit laws

Assume that both these exist,

\[
\lim_{x \to a} f(x), \quad \lim_{x \to a} g(x),
\]

then the following hold:

3. The limit of \( cf(x) \) at \( x = a \) exists for any \( c \) and

\[
\lim_{x \to a} [cf(x)] = c \left( \lim_{x \to a} f(x) \right)
\]

4. The limit of \( f(x)g(x) \) at \( x = a \) exists and

\[
\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)
\]
Limit laws

Assume that both these exist,

\[ \lim_{x \to a} f(x), \quad \lim_{x \to a} g(x), \]

then the following hold:

5. The limit of \( f(x)/g(x) \) at \( x = a \) exists for any \( c \) and

\[ \lim_{x \to a} \left[ f(x)/g(x) \right] = \left( \lim_{x \to a} f(x) \right) / \left( \lim_{x \to a} g(x) \right) \]

so long as \( \lim_{x \to a} g(x) \neq 0 \).
Assume that both these exist,

\[ \lim_{x \to a} f(x), \quad \lim_{x \to a} g(x), \]

then the following hold:

6. The limit of \( f(x)^n \) at \( x = a \) exists for any natural number \( n \) and

\[ \lim_{x \to a} [f(x)^n] = \left( \lim_{x \to a} f(x) \right)^n \]
Limit laws

Assume that both these exist,

$$\lim_{x \to a} f(x), \quad \lim_{x \to a} g(x),$$

then the following hold:

7. The limit of $f(x)^{n/m}$ at $x = a$ exists for any natural number $n$ and

$$\lim_{x \to a} \left[f(x)^{n/m}\right] = \left(\lim_{x \to a} f(x)\right)^{n/m}$$

so long as $f(x) \geq 0$ near $x = a$ and $m, n$ have no common factors.
Question

Does the limit

\[ f(x) = |x| \]

at \( x = 0 \) exist?
Question

Does the limit of these functions

\[ f(x) = x, \quad g(x) = x^2, \]

at \( x = 0 \) exist?
Question

Does the limit of these functions

\[ f(x) = \frac{x^2}{|x|}, \quad g(x) = \frac{x}{|x|} \]

at \( x = 0 \) exist?
Limit laws

Assume that both these exist,

\[
\lim_{x \to a} f(x), \quad \lim_{x \to a} g(x),
\]

then the following hold:

1. The limit of \( f(x) + g(x) \) at \( x = a \) exists and

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

2. The limit of \( f(x) - g(x) \) at \( x = a \) exists and

\[
\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
\]
Limit laws

Assume that both these exist,

\[ \lim_{x \to a} f(x), \quad \lim_{x \to a} g(x), \]

then the following hold:

3. The limit of \( cf(x) \) at \( x = a \) exists for any \( c \) and

\[ \lim_{x \to a} [cf(x)] = c \left( \lim_{x \to a} f(x) \right) \]

4. The limit of \( f(x)g(x) \) at \( x = a \) exists and

\[ \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \]
Limit laws

Assume that both these exist,

$$\lim_{x \to a} f(x), \quad \lim_{x \to a} g(x),$$

then the following hold:

5. The limit of $f(x)/g(x)$ at $x = a$ exists for any $c$ and

$$\lim_{x \to a} \left[ f(x)/g(x) \right] = \left( \lim_{x \to a} f(x) \right) / \left( \lim_{x \to a} g(x) \right)$$

so long as $\lim_{x \to a} g(x) \neq 0$. 
Continuous functions

If \( f(x) \) and \( g(x) \) are continuous at \( x = a \), is \( f(x) + g(x) \) continuous at \( x = a \)?
Continuous functions

If \( f(x) \) and \( g(x) \) are continuous at \( x = a \), is \( f(x) + g(x) \) continuous at \( x = a \)?

What about continuous in general?
Polynomial functions

Because $f(x) = x$ is a linear function, using rule 4 then

$$\lim_{x \to a} x^n = \left( \lim_{x \to a} x \right)^n = a^n,$$

using rule 2 and rule 1 then if

$$p(x) = c_0 + c_1x + \ldots + c_nx^n$$

then

$$\lim_{x \to a} p(x) = c_0 + c_1a + \ldots + c_na^n = p(a).$$
Example

What is the limit as

$$\lim_{x \to 5} x^3 - 20x + 3 =$$
Questions