What did we do last lecture?

14th of September 2016
Calculating the tangent

The tangent line is what a curve looks like as we zoom in:
Calculating the tangent

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The tangent line is what a curve looks like as we zoom in:

1x

5x
Calculating the tangent

The tangent line is what a curve looks like as we zoom in:

\[ 1x \quad \text{and} \quad 5x \]
Calculating the tangent

The tangent line is what a curve looks like as we zoom in:
Calculating the tangent

The tangent line is what a curve looks like as we zoom in:

The red line is the tangent line.

The red line is the tangent line.
Calculating the tangent

The slope of any secant between \( a \) and \( x \) is given by

\[
m_{x,a} = \frac{\text{rise}}{\text{run}} = \frac{f(x) - f(a)}{x - a}.
\]
Calculating the tangent

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The slope of the tangent, $m_t$, is what happens when $x$ approaches $a.$
Calculating the tangent

One way of doing this is

\[ m_t = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \]

This is precisely the same as letting \( x = a + h \) and letting \( h \to 0 \). So

\[ m_t = \lim_{h \to 0} \frac{f(a + h) - f(a)}{(a + h) - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \]

so that \( m_t \) is the slope of the tangent line at \((a, f(a))\).
Calculating the tangent

So the slope of the tangent

\[ m_t = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} . \]

The tangent at \((a, f(a))\) is the line of slope \(m_t\) through \((a, f(a))\):

\[ y - f(a) = m_t(x - a) \]

so let's put it together in an example.
Calculating the tangent

Let us take an example,

\[ f(x) = \frac{1}{x}, \]

let us calculate the tangent at \( x = 2 \).
Calculating the tangent
So firstly let us calculate the slope

\[ f(x) = \frac{1}{x}, \]

at \((2, 1/2)\). So we need the limit

\[ m_t = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{1}{2 + h} - \frac{1}{2}. \]
Calculating the tangent

Using the value $m_t = 1/4$, the tangent at $(x_0, y_0) = (2, 1/2)$

\[
y - y_0 = m_t(x - x_0), \quad y - \frac{1}{2} = -\frac{1}{4}(x - 2), \quad y = -\frac{1}{4}x + 1
\]
Calculating the tangent

Using the value $m_t = 1/4$, the tangent at $(x_0, y_0) = (2, 1/2)$

\[ y - y_0 = m_t(x - x_0), \]
\[ y - 1/2 = -1/4(x - 2), \quad y = -1/4x + 1 \]
Constructing a tangent line

Rate of change

Given a function $f$, the average rate of change on the interval $[a, x]$ is

$$m_s = \frac{f(x) - f(a)}{x - a},$$

the instantaneous rate of change at $x = a$ is

$$m_t = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

which is the slope of the tangent line, provided it exists.
Constructing a tangent line

Given a function \( f \), the tangent at \( x = a \) is the unique line through \( (a, f(a)) \) with a slope equal to the instantaneous rate of change at \( x = a \), \( m_t \), its equation is

\[
y - f(a) = m_t(x - a),
\]
Constructing a tangent line

Lets do an example in the book

\[ f(x) = x^3 + 4x \]

at \( x = 1 \) (since \( f(1) = 5 \), we are going through the point \((1, 5)\)).

\[ m_t = \]
Constructing a tangent line

Lets do an example in the book

\[ f(x) = x^3 + 4x \]

at \( x = 1 \) (since \( f(1) = 5 \), we are going through the point \( (1, 5) \)).

\[ m_t = 7, \]
The Derivative

The derivative is a **function** representing the slope of the tangent at any point.

Given any \( x \), the derivative is the function

\[
f'(x) = \frac{df(x)}{dx} = \lim_{x \to a} \frac{f(x + h) - f(x)}{h},
\]

provided it exists. If this limit exists at a point \( x = a \) we call the function differentiable at \( x = a \).
The tangent again

To find the slope of the tangent,

\[ m_t = f'(a) \]

so that the general formula for the tangent for \( x = a \), at the point \((a, f(a))\) is

\[ y - f(a) = f'(a)(x - a). \]
The Derivative

Let us consider the curve

\[ y = f(x) = \sin(x), \]

the derivative (as we will see later) is \( \cos(x) \).

Notice the slope of the tangent is 0 when \( \cos(x) = 0 \).
Higher derivatives

We can certainly take derivatives of derivatives.

### Higher order derivatives

Assuming that \( f \) can be differentiated as often as necessary, we define

\[
f''(x) = \frac{df'(x)}{dx} = \frac{d^2f(x)}{dx^2},
\]

more generally, for any integer \( n \geq 1 \), the \( n \)-th derivative is

\[
f^{(n)}(x) = \frac{d^n f(x)}{dx^n}, \quad \frac{df^{(n-1)}(x)}{dx^2}.
\]
Let us compute the derivative of 

\[ f(x) = x^5 \]
Polynomials

More generally

\[ f(x) = x^n \]
If $n$ is any integer and $f(x) = x^n$, then the derivative is

$$f'(x) = \frac{df(x)}{dx} = nx^{n-1}.$$ 

This applies to polynomials ONLY.
Polynomials

Using this general rule, we find the derivative is

\[ f(x) = x^4 + 2x^2 + 2, \]

hence, we can calculate the tangent at \( x = 1 \) as follows.