Week 4, Recitation 1

1. Let \( f(x) = x^2 \)
   (a) Find the equation of the tangent lines to the graph of \( f(x) \) at \((1,1)\) and at \((a,a^2)\).
   (b) Find the x-intercept of the tangent line to \( f(x) \) at \( x = a \). Sketch \( f(x) \) and the tangent line.
   (c) Find the y-intercept of the tangent line to \( f(x) \) at \( x = a \).
   (d) What are the closest points on the graph of \( f(x) \) to the point \((0,b)\)?
   (e) Which points on the graph of \( f \) have a tangent line that pass through \((0,-9)\)?

2. Find the tangent line to the curve \( f(x) = 2\sin x + 3\cos x + 5\tan x \) at \( x = \pi/6 \).

3. Show that no line tangent to the graph of \( f(x) = x + \frac{1}{x} \) passes through the origin.

4. Which value(s) of \( a \) make the function \( f(x) \) continuous? differentiable?

   \[
   f(x) = \begin{cases} 
   ax^2 & \text{for } x \geq 2 \\
   \frac{x^2}{x^2} + 3 & \text{for } x < 2 
   \end{cases}
   \]

5. Let \( f(x) = x^2 + c \).
   (a) For what values of \( c \) does the tangent line to the graph of \( f(x) \) at \( x = 1 \) pass through \((3,5)\)?
   (b) For what values of \( c \) does the normal line to the graph of \( f(x) \) at \( x = 2 \) pass through \((8,1)\)?

6. If \( g(x) = f(x) + c \), then \( g'(x) = f'(x) \);

7. If \( g(x) = cf(x) \), then \( g'(x) = cf'(x) \).

8. (a) If \( f(x) = u(x)v(x)w(x) \), find a formula for \( f'(x) \).
   (b) Find \( f'(1) \), where \( f(x) = (x^3 + 3x^2 - 6x + 1)(x^4 - x^2 - 1)(\sqrt{x}) \).

9. Suppose \( g, u, \) and \( v \) are functions defined for all real numbers. Find a formula for \( f'(x) \) where \( f \) is defined by
   (a) \( f(x) = g(u(v(x))) \)
   (b) \( f(x) = g(u(x) + v(x)) \)
   (c) \( f(x) = g(u(x)v(x)) \)
   (d) \( f(x) = g(u(x)/v(x)) \)

10. Compute \( g'(1) \), where \( g(x) = \frac{x+f(x)}{x-f(x)} \), \( f(1) = 4 \), and \( f'(1) = 2 \).

11. Suppose \( f(x) = xf'(x) \) for all \( x \) in the domain of \( f(x) \) then what can we say about the function \( g(x) \)?

12. Fill in the table, given that \( h(x) = f(g(x)) \).

<table>
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<th>( a )</th>
<th>( g(a) )</th>
<th>( g'(a) )</th>
<th>( f(a) )</th>
<th>( f'(a) )</th>
<th>( h(a) )</th>
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</table>

13. Suppose \( f(x) \) is a function that is differentiable for all real \( x \), and \( f(a+b) = f(a)f(b) \) for all real \( a \) and \( b \) with \( f'(0) = 1 \) and \( f(0) = 1 \).
   (a) Show that \( f(x) = f'(x) \) for all real \( x \).
   (b) Draw a picture of such a function \( f(x) \).
   (c) Suppose we don’t know that \( f'(0) = 1 \). What can you say about \( f(x) \) and \( f'(x) \)?