1 Functions

Some folks are confused by functions. This confusion is in part a byproduct of the equals sign (=) which is used for function definition in algebra. This equals sign simply is a shorthand way of declaring the function name and its parameters.

For example, let’s consider the function:

$$f(x) = 5x + 1.$$ 

Here our function name is $f$ and it takes a parameter $x$. The equation $5x + 1$ is the operation the function performs on $x$.

So when we input a ‘value’ into the function say for instance 7, we replace all occurrences of $x$ with that value.

Our equation thus becomes:

$$f(7) = 5 \cdot 7 + 1 = 36.$$ 

This is a way of saying that when we input the value of 7 into our function, it produces an output value of 36.

Our function can take any ‘value’ we like. What if we want to input something nonsensical like $cows$?

Then our equation becomes:

$$f(cows) = 5 \cdot cows + 1.$$ 

Here our function took the input value of $cows$ and produced the output value of $5 \cdot cows + 1$. What happens if we let $cows = 10$? Then we replaces all occurrences of $cows$ with the numerical value of 10 and the value of the equation becomes 51.

Our function can even take the input of another function, even itself!

$$f(f(x)) = 5f(x) + 1 = 5 \cdot (5x + 1) + 1 = 25x + 6.$$ 

This abstraction can be difficult for some folks to wrap their heads around. If you are having trouble understanding this concept, then you MUST seek help immediately! If you don’t, then your chances of passing this class are reduced to nearly zero.

1.1 Function Naming

In mathematics we like to call functions using short names like $f$, $g$, or $h$, which is why you so often see them labeled as such. We also like to use short variable names like $x$, $y$ and $z$, but these are all simply names. These conventions are used to help keep everyone on the same page when working out problems. However, we are not limited to these names! We could for instance make a function called $SquareArea$ which has a parameter of $length$. This function would look something like:

$$SquareArea(length) = length \cdot length.$$ 

You might see the above equation shortened to:

$$A_{\text{square}}(l) = l^2.$$ 

Both of these equations have the same meaning and perform the same operations! They are indeed equivalent functions.

1.2 Sloppy Math

Sometimes we get a little careless when we are working out math. We will drop the functions parameters while simplifying equations. I am sure you have seen instances where $sin(x)$ is simply called $sin$. This is a case where we have dropped the parameter to make working out the algebra a little less writing intensive. This same philosophy is often applied to other functions like our old pal $f(x)$. In the near future, you might see something along the lines of:

$$\frac{df}{dx}$$

This is a shorthand way of saying the derivative of the function $f(x)$ with respect to the variable $x$. Make no mistake, this is still a function and takes a parameter $x$. 
1.3 Composition of Functions

As discussed briefly above, a function can take any ‘value’ we like. This includes other functions. When a function takes an input ‘value’ of another function this is called function composition.

Say we have two functions:

\[ f(x) = 5x + 1 \]
\[ g(x) = x^2 + 4 \]

When \( g(x) \) takes the value of \( f(x) \), we say “\( g \) composed with \( f \) of \( x \)” and write it as:

\[
(g \circ f)(x) = g(f(x)) \\
= (f(x))^2 + 4 \\
= (5x + 1)^2 + 4 \\
= 25x^2 + 10x + 1 + 4 \\
= 25x^2 + 10x + 5.
\]

Likewise when \( f(x) \) takes the value of \( g(x) \) it is we say “\( f \) composed with \( g \) of \( x \)” and write it as:

\[
(f \circ g)(x) = f(g(x)) \\
= 5 \cdot g(x) + 1 \\
= 5 \cdot (x^2 + 4) + 1 \\
= 5x^2 + 21.
\]

In both cases we are defining a new function which is called \( (f \circ g)(x) \) or \( (g \circ f)(x) \) with the parameter named \( x \).

1.4 Exponents

Some rules for exponents:

<table>
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<tr>
<th>Product Rules</th>
<th>( x^a y^a = (xy)^a )</th>
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<tr>
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<td>( x^{p+q} = x^p x^q )</td>
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<tr>
<td>Quotient Rules</td>
<td>( \frac{x^p}{x^q} = x^{p-q} )</td>
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<tr>
<td></td>
<td>( \frac{x^a}{y^b} = \left( \frac{x}{y} \right)^a )</td>
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<td>Power Rules</td>
<td>( (x^a)^b = x^{ab} )</td>
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<td>( x^{\frac{1}{a}} = \sqrt[a]{x} )</td>
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<td>( x^{\frac{m}{n}} = \sqrt[n]{x^m} )</td>
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<td>( x^0 = 1 )</td>
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<tr>
<td></td>
<td>( 0^q = 0 ) for all ( q &gt; 0 ).</td>
</tr>
</tbody>
</table>

Consider the Product Rule for a moment. Some folks may be confused by this concept, but you can think of it like this:

\[
f(x) = x^2 x^3 = (xx)(xxx) = x^5 \]
\[
f(x) = x^2 x^{-3} = \frac{x x}{xxx} = x^{-1}
\]

Notice how on the first line there are 5 \( x \)'s being multiples together? That’s what the 5\(^{th} \) power denotes. Notice on the second line that there is one fewer \( x \) in the numerator than the denominator? That’s what the \(-1\) power denotes.
1.5 Logarithms

Many people groan when they hear the word ‘logarithm.’ However, logarithms are amazing functions which can be used to turn multiplication and division into addition and subtraction!

Logarithm Product Rule
\[ \log_b(xy) = \log_b x + \log_b y \]

Logarithm Quotient Rule
\[ \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \]

Logarithm Power Rule
\[ \log_b(x^q) = q \log_b x \]

Logarithm base change Rule
\[ \log_b a = \frac{\log_c a}{\log_c b} \]

Logarithm Raised to its Base
\[ b^{\log_b x} = x \]

Logarithm of the base
\[ \log_b b = 1 \]

Logarithm of 0
- Undefined

Logarithm of 1
\[ \log_b 1 = 0 \]

Logarithm of negative
- Undefined, for all \( x < 0 \)

Some examples of the above log rules.

- Logarithm Product Rule
  \[ \log_{10}(25) = \log_{10}5 + \log_{10}5 \]

- Logarithm Quotient Rule
  \[ \log_{10} \left( \frac{\frac{3}{7}}{5} \right) = \log_{10}3 - \log_{10}5 \]

- Logarithm Power Rule
  \[ \log_{10}(16) = \log_{10}4^2 = 2\log_{10}4 \]

- Logarithm base change Rule
  \[ \log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)} = 1.77 \]

- Logarithm Raised to its Base
  \[ y^{\log_{10}x} = x \]
  \[ y^{\ln(x)} = x \]

- Logarithm of the Base
  \[ \log_{10}(10) = 1 \]
  \[ \ln(e) = 1 \]

2 Slope

A slope, \( m \), is the change in \( y \) divided by the change in \( x \). This is a core concept of calculus and will be expanded upon in the future.

\[
\text{Slope} = m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}
\]

(1)

Special note: \( \frac{dy}{dx} \) is the instantaneous rate of change. Limits are used to compute this value, but we will get to that later.

Figure 1: Example of rise and run of a line.

We compute the slope of this line as follows:

\[
\text{Slope} = \frac{3 - 0}{5 - 1} = \frac{3}{4}
\]
Take a moment and compute the following slopes:

To find the slope we need to compute
\[
\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}.
\]
Where \(y_2 = f(x), y_1 = f(a), x_1 = a,\) and \(x_2 = x\). Plugging in we get:
\[
\text{Slope} = m = \frac{f(x) - f(a)}{x - a}.
\]

We can follow similar steps to solve the slope of (b). From the image we are given that \(y_2 = f(x + h), y_1 = f(x), x_2 = x + h,\) and \(x_1 = x\). Thus
\[
m = \frac{f(x + h) - f(x)}{x + h - x} = \frac{f(x + h) - f(x)}{h}.
\]

### 3 Lines

Next we will review the process of drawing a line using an equation.
As you might recall, the function for a straight line, \(f(x)\), is defined as
\[
f(x) = mx + b, \tag{2}
\]
where \(b\) is the y-intercept, \(m\) is the slope, and \(x\) is the function argument.
Note: To compute the value of \(y\) at a single point \(x\), we use the equation:
\[
y = mx + b.
\]

Use the empty graphs to graph the following equations:

- \(f(x) = -\frac{1}{2}x + 2\)
- \(g(x) = \frac{1}{2}x + 1\)
3.1 Lines through a point

We can also find the equation of a line given a point \((x_1, y_1)\) with a slope \(m\) using the equation:

\[ y - y_1 = m(x - x_1). \]

Find the equation of a line that passes through the following points

(a) \((2, 2), \ m = 4\)
(b) \((-2, 3), \ m = -2\)
(c) \((4, 1), \ m = -\frac{3}{4}\)

Plugging in for (a), we get the following:

\[ y - 2 = 4(x - 2) \]
\[ y - 2 = 4x - 8 \]
\[ y = 4x - 8 + 2 \]
\[ y = 4x - 6 \]

Plugging in for (b), we get the following:

\[ y - 3 = -2(x - (-2)) \]
\[ y - 3 = -2(x + 2) \]
\[ y - 3 = -2x - 4 \]
\[ y = -2x - 4 + 3 \]
\[ y = -2x - 1 \]
Finally for (c) we get the following:

\[ y - 1 = \frac{-3}{4}(x - 4) \]
\[ y - 1 = \frac{-3}{4}x + 3 \]
\[ y = \frac{-3}{4}x + 3 + 1 \]
\[ y = \frac{-3}{4}x + 4 \]

3.2 Horizontal and Vertical Lines

A vertical line is a straight line such that for every \( y \) the corresponding \( x \) value is the same. This means that a vertical line is not a function. A side effect of the line being vertical is that the slope is undefined.

Given a horizontal line, each \( x \) value has the same \( y \) value. The slope of a horizontal line is 0.

Figure 3: Examples of horizontal and vertical lines.

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3.3 Tangent lines

A tangent line is a line which touches a curve at a single point, without cutting across the curve and is perpendicular to the curve at that point. A line which is tangent to a curve has the same slope as the curve at this point. This is an important concept that will be discussed later!

(a) Line tangent to \( \sin(x) \) at \( x = 1 \)  
(b) Line tangent to \( \sin(x) \) at \( x = \pi \)  
(c) Line tangent to \( \sin(x) \) at \( x = \frac{\pi}{2} \)

Figure 4: Examples of tangent lines.

3.4 Graphs to know by heart!

(a) \( f(x) = x \)  
(b) \( f(x) = \frac{1}{2} \)  
(c) \( f(x) = x^2 \)  
(d) \( f(x) = \frac{1}{x^2} \)  
(e) \( f(x) = x^3 \)  
(f) \( f(x) = \sqrt{x} \)  
(g) \( f(x) = |x| \)  
(h) \( f(x) = \ln(|x|) \)  
(i) \( f(x) = e^x \)

Figure 5: Graphs to know.
3.5 Shifting Graphs

If you have a function, then you can 'shift' that function by modifying a few parameters. A general way of looking at this is with the following function definition:

\[ f(x) = A(x - H)^q + V, \]  

(3)

where \( A \) is the amplitude or scale of the function, \( H \) is the horizontal offset, and \( V \) is the vertical offset. Take for example the function \( f(x) = x^3 \) (Figure 5e).

(a) \( f(x) = (x - 2)^3 \) where \( A = 1, H = 2, V = 0 \)

(b) \( f(x) = (x + 2)^3 \) where \( A = 1, H = -2, V = 0 \)

(c) \( f(x) = x^3 - 2 \) where \( A = 1, H = 0, V = -2 \)

(d) \( f(x) = x^3 + 2 \) where \( A = 1, H = 0, V = 2 \)

(e) \( f(x) = -x^3 \) where \( A = -1, H = 0, V = 0 \)

Try and graph the following functions:

(a) \( f(x) = -|x + 3| + 4 \)

(b) \( g(x) = (x + 3)^2 - 3 \)

(c) \( h(x) = \frac{1}{x+2} \)
You should have come up with graphs that look like the following:

![Graphs](https://via.placeholder.com/150)

(f) $-|x + 3| + 4$

(g) $(x + 3)^2 - 3$

(h) $\frac{1}{(x+2)}$

Figure 6: Graph shifting answers.

## 4 Piecewise Functions

A piecewise function is a function which is composed of multiple ‘pieces’. In other words, it is a function that is made up of parts of other functions, which are defined over their own intervals. For example, consider these two piecewise functions:

\[
\begin{align*}
  f(x) &= \begin{cases} 
  |x| & x < 3 \\
  -x^2 & x \geq 3 
  \end{cases} \\
  g(x) &= \begin{cases} 
  2x & x < 1 \\
  2x^2 & x > 1 \\
  0 & x = 1 
  \end{cases}
\end{align*}
\]

![Graphs](https://via.placeholder.com/150)

(a) $f(x)$

(b) $g(x)$

Figure 7: Examples of piecewise functions.

Take a moment and graph the following piecewise function:

\[
h(x) = \begin{cases} 
  x + 2 & x < 1 \\
  (x - 1)^2 & x \geq 1 
  \end{cases}
\]
5 Increasing and Decreasing

- A function is said to be increasing on an interval if for all $x_1 < x_2$ then $f(x_1) \leq f(x_2)$.
- A function is said to be strictly increasing on an interval if for all $x_1 < x_2$ then $f(x_1) < f(x_2)$.
- A function is said to be decreasing on an interval if for all $x_1 > x_2$ then $f(x_1) \leq f(x_2)$.
- A function is said to be strictly decreasing on an interval if for all $x_1 > x_2$ then $f(x_1) < f(x_2)$.

Here we can see that the function in Figure 9 is increasing on the interval $(-\infty, -2) \cup (1, \infty)$ (positive slope), decreasing on $(-2, 1)$ (negative slope), and neither increasing nor decreasing at $x = -2$ and $x = 1$ (zero slope).
6 Finding Vertical Asymptotes

To find a vertical asymptote, set the denominator of the function to zero and solve.
Take for example \( f(x) = \frac{1}{x} \). Here we have a vertical asymptote at \( x = 0 \). This is where the function becomes undefined.
Next we will find the vertical asymptotes of the function \( g(x) = \frac{1}{x^2 - 1} \).

![Graph of \( g(x) = \frac{1}{x^2 - 1} \).](image)

From a visual inspection of the graph, we can see that the vertical asymptotes at \( x = -1 \) and \( x = 1 \). Now let’s see if we can mathematically solve for the vertical asymptotes.

\[
x^2 - 1 = 0
\]
\[
(x - 1)(x + 1) = 0
\]
\[
x - 1 = 0 \text{ or } x + 1 = -1
\]
\[
x = 1 \text{ or } x = -1
\]

Take a moment and find the vertical asymptotes for:
(a) \( \frac{1}{x^2 - 1} \)
(b) \( \frac{1}{x^2 + x - 6} \)
For (a):

\[ x^2 - 4 = 0 \]
\[ (x - 2)(x + 2) = 0 \]
\[ x - 2 = 0 \text{ or } x + 2 = 0 \]
\[ x = 2 \text{ or } x = -2 \]

For (b):

\[ x^2 + x - 6 = 0 \]
\[ (x - 2)(x + 3) = 0 \]
\[ x - 2 = 0 \text{ or } x + 3 = 0 \]
\[ x = 2 \text{ or } x = -3 \]

## 7 Finding Horizontal Asymptotes

Suppose we have an \( n^{th} \) degree polynomial being divided by a \( m^{th} \) degree polynomial.

\[
f(x) = \frac{a_1 x^n + a_2 x^{n-1} + \ldots + a_n x^0}{b_1 x^m + b_2 x^{m-1} + \ldots + b_m x^0}
\]

- If \( n < m \) then the \( x \)-axis is the horizontal asymptote.
- If \( n = m \) then the line \( y = \frac{a_1}{b_1} \) is the horizontal asymptote.
- If \( n > m \) then there is no horizontal asymptote.

Let’s consider the following functions:

\[
g(x) = \frac{3x^2 + 2x}{5x^3 + 3} \\
h(x) = \frac{3x^3 + 4x^3}{7x^3 + 2} \\
p(x) = \frac{6x^3}{3x}
\]

- \( g(x) \) has a horizontal asymptote along the \( x \)-axis.
- \( h(x) \) has a horizontal asymptote at \( y = \frac{4}{7} \).
- \( p(x) \) has no horizontal asymptote.

Try and see if you can find the horizontal asymptotes for:

(a) \( f(x) = \frac{3x^2 + 2x^4}{4x^8} \)

(b) \( g(x) = 7x^2 \)

(c) \( h(x) = \frac{7x^3 + 8x^2}{4x^7 + 3x + 2x^3} \)

For (a), \( m = 6 \) and \( n = 4 \) so \( n < m \). This means that the \( x \)-axis is the horizontal asymptote.

For (b), \( m = 0 \) and \( n = 2 \) so \( n > m \). This means that there is not horizontal asymptote.

For (c), \( m = 3 \) and \( n = 3 \) so \( n = m \). This means that \( y = 7/2 \). Don’t get confused if the highest exponent isn’t listed first!